Novembre 2018

Extracting decision rules from qualitative data using generalised qualitative integrals.

1 abstract

This TER is based on the papers [1, 2]. In the first, one the Sugeno integral is encoded as decision rules. The second one presents some generalisations of Sugeno integrals changing the implication used in the definitions. More precisely the Kleene Deenes implication is replaced by the Godel implication or the contraposed Godel implication. In this TER we only study the case of the Godel implication.

The aim of this work is the encoding of the generalised Sugeno integrals using the godel implication as decision rules.

2 Discrete Sugeno integral

The discrete Sugeno integral is used in qualitative decision theory and multiple criteria decision making where objects x or alternatives are evaluated according to criteria. The set of criteria is denoted $\mathcal{C} = \{1, \cdots, n\}$. The evaluation scale L is the same for all the criteria and for the global evaluation. We assume that L is a totally oredered set with a top element denoted 1 and a bottom element denoted 0. For example we can consider the real intervall $[0, 1]$.

Example 1. *Students are evaluated according to the following criteria :* $\mathcal{C} = \{maths, english, physics\}$. *The evaluation scale is* $L = \{0, 0.25, 0.5, 0.75, 1\}$. *By example, an etudiant* e *can obtain* 0.25 *in maths,* 1 *in english and* 0 *in physics. Hence* e *is represented by the vector* $(0.25, 1, 0)$.

The Sugeno integral synthesizes the partial ratings of an object into a global evaluation. Its definition is based on a fuzzy measure representing the importance, weight of the criteria coalition.

Definition 1. *A fuzzy measure is a set fonction* $\mu : 2^{\mathcal{C}} \to L$ *such that* $\mu(T) \leq \mu(S)$ *if* $T \subseteq S$, $\mu(\emptyset) = 0$ *and* $\mu(\mathcal{C}) = 1$.

Example 2. *We consider three criteria* $\mathcal{C} = \{1, 2, 3\}$ *and the evaluation scale* $L = \{0, 0.25, 0.5, 0.75, 1\}$. *An example of fuzzy measure is*

	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
μ	0	0.25	0	0.5	0.25	0.75	1	1

All the information of a fuzzy measure is contained in its qualitative Moebius transform defined as follows.

Definition 2. *The qualitative Moebius transform of the fuzzy measure* μ *is a set function* $\mu_\#$ *defined by : forall* $T \subseteq \mathcal{C}$, $\mu_\#(T) = \mu(T)$ *if* $\mu(T) > \max_{S \subset T} \mu(S)$ *and* 0 *otherwise.*

The set T such that $\mu_\#(T) > 0$ are named the focal sets of μ.

Example 3. *The qualitative Moebius associated to the fuzzy measure* μ *presented in the previous example is*

	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mu_\#$	0	0.25	0	0.5	0	0.75	1	0

The focal sets are $\{1\}, \{3\}, \{1,3\}, \{2,3\}$.

We have for all $T \subseteq \mathcal{C}$, $\mu(T) = \max_{S \subset T} \mu_\#(S)$.

Definition 3. *The Sugeno integral of an object* $x = (x_1, \cdots, x_n)$ *with respect to the fuzzy measure* μ *is*

$S_\mu(x) = \max_{\alpha \in L} \min(\alpha, \mu(X_\alpha))$ *where* $X_\alpha = \{i \in \mathcal{C} | x_i \geq \alpha\}$.

We have equivalent expressions :

$$S_\mu(x) = \max_{T \in \mathcal{C}} \min(\mu(T), \min_{i \in T} x_i) = \max_{T \in \mathcal{C}} \min(\mu(T), \min_{i \in T} x_i).$$

The qualitative Moebius transform is enough to calculate Sugen integral, i.e.,

$$S_\mu(x) = \max_{T \in \mathcal{C}} \min(\mu_\#(T), \min_{i \in T} x_i) = \max_{T \in \mathcal{C}} \min(\mu_\#(T), \min_{i \in T} x_i).$$

Note that using the qualitative Moebius transform, the complexity of the Sugeno integral is reduced.

Example 4. *We consider the same evaluation scale, the same criteria, the same fuzzy measure and the objects* $x = (0.25, 1, 0)$.

$$\begin{aligned} S_\mu(x) = &\ \max_{T \in \mathcal{C}}[\min(\mu_\#(T), \min_{i \in T} x_i)] \\ = &\ \max(\min(0.25, 0.25), \min(0.5, 0), \min(0.75, 0), \min(1, 0)) \\ = &\ 0.25 \end{aligned}$$

The Sugeno integral is equivalent to a set of $if-then$ rules :

Property 1. *Each focal set T corresponds to the rules :*

for all i in T if $x_i \geq \mu(T)$ then $S_\mu(x) \geq \mu(T)$.

In this case we speak about selection rule. Note that one obtains elimination rules concidering the conjugate capacity (defined by $\mu^c(A) = 1 - \mu(A^c)$ where A^c is the complementary of A) and its focal sets.

Example 5. *The Sugeno integral considered in the previous example corresponds to the following set of selection rules :*

$$\begin{cases} \textit{If } x_1 \geq 0.25 \textit{ then } S_\mu(x) \geq 0.25; \\ \textit{If } x_3 \geq 0.5 \textit{ then } S_\mu(x) \geq 0.5; \\ \textit{If } x_1 \geq 0.75 \textit{ and } x_3 \geq 0.75 \textit{ then } S_\mu(x) \geq 0.75; \\ \textit{If } x_2 \geq 1 \textit{ and } x_3 \geq 1 \textit{ then } S_\mu(x) \geq 1. \end{cases}$$

3 Generalized Sugeno integral

The Sugeno integral is defined by the Kleene Deenes conjonction i.e the minimum. This minimum calculated between $\mu(T)$ and $\min_{i\in T} x_i$, for a set of criteria T, defines the action of the fuzzy measure μ on the partial values before agregating them with a maximum. Replacing this conjunction by another one entails others aggregation functions named qualitative integrals or generalized Sugeno integrals.

In this TER we are going to use

- the Godel implication : $a \to_G b$ equal to 1 if $a \leq b$ and b otherwise ;
- its semi dual conjunction $a \otimes b = 1 - a \to (1 - b)$. We have $a \otimes b = 0$ if $a + b \leq 1$ and b otherwise ;

where a and b are elements in L.

In such a context the qualitative integrals are

$S_\mu^\otimes x = \max_{A\subseteq\mathcal{C}}(\mu(A) \otimes \min_{i\in A} x_i)$ and

$S_\mu^\to x = \min_{A\subseteq\mathcal{C}}(\mu^c(A) \to_G (\max_{i\in A} x_i))$.

The second expression uses the conjugate of the capacity μ denoted μ^c. The conjugate is defined on the scale equipped with a negation. In a first time we consider the real interval $[0, 1]$ and the negation is $1 - ..$ In such a context $\mu^c(A) = 1 - \mu(\bar{A})$ where $\bar{A}$ is the complementary of the set A.

Example 6. *We consider three criteria* $\mathcal{C} = \{1,2,3\}$ *and the evaluation scale* $L = \{0, 0.25, 0.5, 0.75, 1\}$ *and the fuzzy measure* μ *of the example 1. A negation can be defined on* L, $1-0 = 1$,$1-0.25 = 0.75$ $1-0.5 = 0.5$ $1-0.75 = 0.25$ *and* $1-1 = 0$. *By exemple,* $\mu^c(\{1\}) = 1-\mu(\{2,3\}) = 1-1 = 0$. *One obtains*

	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
μ^c	0	0	0.25	0.75	0.5	1	0.75	1

4 Some properties

In this section I give some intuitions of properties. One part of the TER is to proove if these properties are satisfied or not.

L'implication $\rightarrow_G$ is decreasing according to the first argument and increasing according to the second one. When we consider a set of criteria there exists a subset which is a focal set. The capacity is the same but the maximum involved is smaller so

$$S_\mu^{\rightarrow_G}(x) = \min_{A \subseteq C} \mu_\#^C(T) \rightarrow_G \max_{i \in T} x_i.$$

Property 2. $S_\mu^{\rightarrow_G}(x) \geq \alpha$ *if and only if for each focal set* T,
- *either there exists* $x_i \in T$ *such that* $\mu_\#^c(T) \leq x_i$,
- *or there exists* $x_i \in T$ *such that* $x_i \geq \alpha$.

The conjunction $\otimes_G$ is increasing in both sides. Each set contains a focal set, the capacity is the same and the maximum involved is greater so

$$S_\mu^{\otimes_G}(x) = \max_{A \subseteq C} \mu_\#(T) \otimes \min_{i \in T} x_i.$$

Property 3. $S_\mu^{\otimes_G}(x) \leq \alpha$ *if and only if for each focal set* T,
- *either there exists* $x_i \in T$ *such that* $x_i \leq \mu_\#^c(\bar{T})$,
- *or there exists* $x_i \in T$ *such that* $x_i \leq \alpha$.

Let us denote (.) the permutation such that $x_{(1)} \leq \cdots \leq x_{(n)}$ and $A_{(i)} = \{(i), \cdots, (n)\}$.

Proposition 1. $S_\mu^{\otimes} x = \max_{A \subseteq \mathcal{C}}(\mu(A_{(i)}) \otimes x_{(i)}) = \max_\alpha \mu(\{f \geq \alpha\}) \otimes \alpha$.

and

$S_\mu^{\rightarrow} x = \min_{A \subseteq \mathcal{C}}(\mu^c(A^c_{(i+1)}) \rightarrow_G x_{(i)}) = \min_\alpha \mu^c(\{f \leq \alpha\}) \rightarrow_G \alpha$.

Idea of the proof

Let i_0 such that $\min_{i \in A} x_i = x_{(i_0)}$ Hence $A \subseteq A_{(i_0)}$ and is the max $\mu(A) \otimes x_{(i_0)}$ is subsumed by $\mu(A_{(i_0)}) \otimes x_{(i_0)}$.

Let i_0 such that $\max_{i \in A} x_i = x_{(i_0)}$ Hence $A \subseteq A^c_{(i_0+1)}$ and is the min $\mu^c(A) \to x_{(i_0)}$ is subsumed by $\mu^c(A_{(i_0+1)}) \otimes x_{(i_0)}$.

Références

[1] D. Dubois, H. Prade and A. Rico : The logical encoding of Sgeno integrals. Fuzzy Sets and Systems, vol. 241 pp. 61-75 (2014)

[2] Dubois D., Prade H., Rico A., Teheux B. (2016) Generalized Sugeno Integrals. In : Carvalho J., Lesot MJ., Kaymak U., Vieira S., Bouchon-Meunier B., Yager R. (eds) Information Processing and Management of Uncertainty in Knowledge-Based Systems. IPMU 2016. Communications in Computer and Information Science, vol 610. Springer.