Reference-dependent Qualitative Models for Decision Making under Uncertainty

Abstract. The aim of this paper is to introduce and investigate a new family of purely qualitative models for decision making under uncertainty. Such models do not require any numerical representation and rely only on the definition of a preference relation over consequences and a relative likelihood relation on the set of events. Within this family, we focus on decision rules using reference levels in the comparison of acts. We investigate both the descriptive potential of such rules and their axiomatic foundations. We introduce in a Savage-like framework, a new axiom requiring that the Decision Maker’s preference between two acts depends on the respective positions of their consequences relatively to reference levels. Under this assumption we determine the only possible form of the decision rule and characterize some particular instances of this rule under transitivity constraints.

1 INTRODUCTION

In the past decade, Decision Making under uncertainty has received much attention in Artificial Intelligence. This is also a traditional topic in Economics where several important results have been published, justifying the use of various criteria, for the comparison of actions under risk and uncertainty. The multiplicity of studies and models developed is easily explainable by the variability of contexts for decision making and the diversity of decision making behaviors we want to describe, explain or simulate. In Decision Theory, axiomatic analysis began with the seminal work of Von Neumann and Morgenstern [16] and Savage [17] providing the foundations of the expected utility model (EU). Despite its normative appeal, the descriptive limits of EU have led mathematical economists to study alternatives to Savage’s theory and to propose various sophisticated extensions of EU, e.g. non-additive integrals like CEU [18]. In this direction, most decision models rely on a cardinal numerical representation of preference (utilities) and uncertainty (probabilities, belief functions and other capacities). Besides this very active current, a parallel stream steadily develops in AI, with a different focus. Indeed, the need of expressive languages for preference handling and reasoning as well as the aim of developing autonomous decision-making agents in real contexts (with poor information) has led researchers to investigate qualitative models. Although less ambitious in their descriptive and prescriptive objectives, they are more operational in practice because they require less information about preference and likelihood of events. In this direction, several qualitative alternatives to EU have been investigated [2, 3, 7, 15, 8, 13, 14, 19].

As pointed out in [4], most of these models, quantitative or qualitative, rely on a numerical representation where utility and uncertainty are commensurate. To escape this assumption which is not always seen as natural, recent papers (see e.g. [5, 4]) investigate an alternative approach based on purely qualitative models, i.e. models that only require the definition of a preference order over consequences and a relative likelihood relation on the set of events. This is the case of Likely Dominance Rules [6, 4] that have been recently introduced in AI and characterized by an Ordinal Invariance Axiom (OI) stating that preferences between two acts only depends on the relative position of their consequences for each state. Unfortunately such models do not offer much flexibility to describe beliefs and preferences in practice. On one hand, due to OI they satisfy (as EU) Savage’s “Sure Thing Principle” which is sometimes contradicted by observed decision making behaviors. On the other hand, the needs to compose purely ordinal and non-commensurate information (likelihood of events, preferences over consequences) induces a conflict between descriptive and normative objectives. Either preferences rely on a coarse belief structure induced by the existence of a predominant events masking any other event in the analysis or they violate minimal transitivity properties required to justify choices or rankings over acts (this is a consequence of Arrow’s theorem, see [1, 4]). To overcome these descriptive and prescriptive limitations, we investigate in this paper an alternative framework (escaping OI) for purely qualitative models. It is based on the introduction of an axiom requiring that preferences between acts depend on the respective position of their consequences (for each state) relatively to reference levels.

The paper is organized as follows: in Section 2, we provide various simple examples showing the descriptive potential of qualitative models based on reference levels. Then, we introduce in Section 3 an axiomatic framework for reference-dependent qualitative models; we determine the possible forms of admissible decision rules. Finally we characterize in Section 4 some particular instance of these rules and discuss the results in the light or Arrow’s theorem and extensions.

2 REFERENCE-DEPENDENT PREFERENCES

2.1 Preliminary definitions and notations

We consider here decision-making problem under uncertainty characterized by a 4-tuple $\langle S, X, A, \succ \rangle$, where $S$ is a finite set of the possible states of nature, $X$ is the set of the possible consequences of acts, $A = X^S$ is the set of potential acts, that is, the set of functions $f : S \rightarrow X$, and $\succ$ is a preference relation on $A$. Within $A$ there exist acts leading to the same consequence for all states in $S$. These acts are called constant acts and denoted $f_x$ for any $x \in X$. For any preference relation $\succeq$, its asymmetric part defines a strict preference denoted $\succ$ and defined by: $x \succ y \Leftrightarrow (x \succeq y \text{ and } \lnot(y \succeq x))$; its symmetric part defines an indifference relation denoted $\sim$ and defined by $x \sim y \Leftrightarrow (x \succeq y \text{ and } y \succeq x)$. If $\geq$ is transitive, then relations $\succ$ and $\sim$ are transitive. A relation $\succeq$ is said to be quasi-transitive if its asymmetric part $\succ$ is transitive.
Following Savage terminology, we call event any subset $A$ of $S$. For any pair of acts $f$ and $g$, the act $fAg$ is defined by: $fAg(s) = f(s)$ if $s \in A$ and $g(s)$ if $s \notin A$. To simplify notation, whenever $f$ or $g$ is a constant act $f_x$ we write $fAx$ (resp. $xAg$) instead of $fAg_x$ (resp. $f_xAg$). An event $A$ is said to be null if and only if: $\forall f, g, h \in A : fAh \sim gAh$. Any event $A \subseteq S$ able to make a discrimination for at least one pair of acts is not null.

### 2.2 Motivating examples

In order to introduce, by example, likelihood dominance rules as well as reference-dependent models, consider the following game:

#### Example 1

Consider a two-player game with 3 fair dice with usual points (see [12]) $f = (1,4,4,4,4,4)$, $g = (3,3,3,3,3,6)$ and $h = (2,2,2,5,5,5)$. Player 1 gets a choice of whichever die he wants, then Player 2 picks one of the remaining dice. They each roll $n$ times, and whoever scores higher the most times, wins. As the second player, which die should we pick?

This can be formalized by considering 8 (virtual) states corresponding to all possible combinations of outcomes, and the following decision table:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>$s_5$</th>
<th>$s_6$</th>
<th>$s_7$</th>
<th>$s_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>5/72</td>
<td>5/72</td>
<td>1/72</td>
<td>1/72</td>
<td>25/72</td>
<td>25/72</td>
<td>25/72</td>
<td>5/72</td>
</tr>
<tr>
<td>$h$</td>
<td>2/72</td>
<td>5/72</td>
<td>2/72</td>
<td>5/72</td>
<td>2/72</td>
<td>5/72</td>
<td>2/72</td>
<td>5/72</td>
</tr>
</tbody>
</table>

Here preferences over dice can be stated by a Likely Dominance Rule (LDR, [4]): $f \succ g \Leftrightarrow \{s \in S, f(s) \geq g(s)\} \supseteq \Lambda \{s \in S, g(s) \geq f(s)\}$ where $\supseteq \Lambda$ is a likelihood relation induced by the probability of events. Such a definition only uses two binary relations ($\supseteq$ and $\supseteq \Lambda$). Note that we have here: $P(f > g) = 50/72$, $P(g > h) = 42/72$, $P(h > f) = 42/72$, which means that $f > g, g > h$ and $f > h$ yielding to intransitive preferences. This gives a winning strategy to Player 2 who can always choose a better die than Player 1. The possibility of describing intransitive preferences is typical of Likely Dominance Rule as defined in [4] to compare acts in decision making problems under uncertainty. A variant of the game consists in changing the rule into: “whoever scores higher or equal than ‘4’ the most times, wins”. Here preferences over dice are defined by: $f \succ g \Leftrightarrow \{s \in S, f(s) \geq 4\} \supseteq \Lambda \{s \in S, g(s) \geq 4\}$. In this case, the introduction of reference level ‘4’ yields a transitive preference order: $f \succ h \succ g$. Such a model is an example of reference-depend LDR we intend to investigate in the paper.

#### Example 2

We consider the “Ellsberg” example [9] with an urn containing blue, red, and green balls. We only know that the proportion of blue balls is $1/3$. The player is invited to pay $5p$ to play, one of the following games $f$ and $g$: he draws a ball then $f$ wins $S_q > 3p$ if the ball is blue and $g$ wins $S_q$ if the ball is red. Then he has a second choice between two other games: $f'$ wins $S_q$ if the ball is blue or green, $g'$ wins $S_q$ if the ball is red or green.

Experiments show that people often prefer $f$ to $g$ (because they are not sure that there is at least $1/3$ of red balls in the urn) but prefer $g'$ to $f'$ (because they are sure there is $2/3$ of red or green balls). Such a decision problem can be formalized as follows: We consider 3 states $s_1$ (blue), $s_2$ (red), $s_3$ (green) and the two possible consequences $x = q - p, y = -p$. The 4 games are represented by acts $f, g, f', g'$ with the following consequences:

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$y$</td>
<td>$x$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$x$</td>
<td>$y$</td>
</tr>
</tbody>
</table>

Such preferences are not compatible with the so-called “Sure-thing Principle” (P2) introduced by Savage [17] in the characterization of EU. This principle states that the preference $f \succ g$ does not depend on states where the two acts have the same consequences. Here this principle is violated here because we have $f > g$ and $g' > f'$, a strict preference reversal only due to a shift from $y$ to $x$ on $s_3$. This reversal makes impossible to represent such preferences by EU or by its qualitative counterparts proposed in [7]. Remark that \{s \in S, f(s) \geq g(s)\} = \{s_1, s_2\} and \{s \in S, g(s) \geq f(s)\} = \{s_2, s_3\}. Then the different status given to pairs $(f, g)$ and $(f', g')$ shows that preferences between two acts do not only depend on the relative position of their consequences for each state. This makes it impossible to represent such preferences with an LDR. However, the introduction of a reference level can easily solve the problem. Indeed preferences can be explained by the following reference-relevant rule:

$f \succ g \Leftrightarrow \{s \in S, f(s) \geq 0\} \supseteq \Lambda \{s \in S, g(s) \geq 0\}$ (1)

with $s_1 \succ s_2$ and $\{s_2, s_3\} \succ \Lambda \{s_1, s_2\}$. These examples show that the introduction of one (or several) reference levels can facilitate the description of observed preferences in some cases. We investigate now in a more formal framework reference-depend ordinal models.

### 3 AN AXIOMATIC FRAMEWORK

#### 3.1 Axioms

We introduce in this section some axioms for reference-dependent preferences. We first review some axioms considered by Savage [1] in the characterization of the EU model and discuss their relevance in our framework. Then we will introduce more specific axioms designed to capture reference-dependent preferences. In the original framework, Savage assumes that Decision Maker’s (DM) preferences over acts have a complete weak order structure (P1 principle). This is necessary to characterize the EU model which requires, by definition, completeness and transitivity of preferences. We consider here a wider class of models including partial preference structures not necessarily transitive. Nevertheless, in the next section we will investigate the impact of transitivity in our framework. Hence, we recall the P1 principle of Savage and introduce a weak version relaxing transitivity for quasi-transitivity:

**Axiom P1.** $\succ$ is a complete weak-order on $A$, i.e., $\succ$ is reflexive, complete and transitive.

**Axiom WP1.** $\succ$ is reflexive and quasi-transitive and its restriction to constant acts is a complete order.

The second axiom of Savage is the so-called “sure-thing principle” (P2). As mentioned above, it requires that the preference between two acts $fAh$ and $gAh$ does not depend on the choice of $h$. This axiom does not necessarily hold when reference levels are used in a decision models. This is quite natural as reference-dependent preferences might be such that $fAh \succ gAh$ and $gAh' \succ fAh'$, as shown by Example 2. In this example indeed, setting $A = \{s_1, s_2\}$ we have $f = fAg_y, g = gAy$, $f' = fax$, and $g' = gAx$. Hence $f > g$ and $g' > f'$ contradict P2. This strict preference reversal would also contradict a weaker version of P2 considered in the axiomatization of qualitative utilities [8].
In order to reveal Decision Maker preferences over certain consequences, the comparison of constant acts is of particular interest. We can indeed consider the following relation:

$$\forall x, y \in X, \quad (x \succeq_X y \iff f_x \succeq f_y)$$

(2)

This definition is self consistent provided that the preference over constant acts should not admit rank reversal of type $f_x A_h \succ f_y A_h$ and $f_x A_h \succ f_y A_h$. This can be captured by the following weakening of the Savage’s P3 postulate:

**Axiom WP3.** $\forall A \subseteq S, \forall h \in A, \quad (x \succeq_X y \iff f_x A_h \succeq f_y A_h)$.

Axiom WP3 is more flexible than P3 (which requires $x \succeq_X y \iff f_x A_h \succeq f_y A_h$) because $x \succ_X y$ is not always sufficient to justify a preference of type $f_x A_h \succ f_y A_h$. For example, when $A$ is a weakly plausible event, we might have: $f_x A_h \sim f_y A_h$ for some $h$.

The fourth principle P4 is introduced by Savage to reveal from $\succ$ a consistent likelihood relation over the set of events. This axiom writes as follows:

**Axiom P4.** $\forall A, B \subseteq S, \forall x, y, x', y' \in X : x \succ_X y$ and $x' \succ_X y'$, $x A_y \succ x B y' \iff x' A_{y'} \succ x' B_{y'}$.

We will not directly use P4 here, as it is a by-product of a stronger axiom introduced later. We introduce now the P5 principle of Savage requiring that preferences over constant acts are not trivial. This axiom writes as follows:

**Axiom P5.** $\exists x, y, z, w \in X s.t. f_x \succ f_y (i.e. x \succ y)$.

Under WP1 and P5 we know that $\succ_X$, the symmetric part of $\succsim_X$ is an equivalence relation with at least two classes. Assuming there exist $q+1$ distinct equivalence classes ($1 \leq q < |X|$), $r_i$, will denote any element of $X$ representing class $i$. In such a way that $f_{r_i} \succ_{X} f_{r_{i+1}}$, $i = 1, \ldots, q$. The set of all the but the worst reference levels is denoted $\mathcal{R} = \{r_1, \ldots, r_q\}$ and represents the various significant separating levels in $X$. In the next section, we will need to consider SP5, a slightly reinforcement of P5 requiring that we can distinguish at least 4 different preference levels on the consequence scale. The axiom writes as follows:

**Axiom SP5.** $\exists x, y, z, w \in X s.t. f_x \succ f_y \succ f_z \succ f_w$. 

We introduce now a more specific axiom stating that preferences over acts only depend on the respective positions of their consequences relatively to these reference levels. To this end, the following reference events have a particular importance:

$$\forall r \in \mathcal{R}, F_r = \{s \in S, f(s) \succeq_X r\}$$

(3)

For acts $g, h, \ldots$ we will use the notations $G_r, H_r, \ldots$, respectively. For the sake of simplicity, event $F_{r_k}$ will be denoted $F_k$, for $k = 1, \ldots, q$. With these notations, the axiom writes:

**Axiom SRL (Separability w.r.t. Reference Levels)** $\forall f, g, f', g' \in A, \forall i, j \in \{1, \ldots, q\}$,

$$F_i = F'_{j}, \quad G_i = G'_{j},$$

$$\forall k \neq i, F_k = G_k \quad \forall k \neq j, F_k = G'_{k} \quad \implies \quad [f \succeq g \iff f' \succeq g']$$

A direct consequence of SRL is the following:

**Axiom DRE (Dependence w.r.t. Reference Event)**

$$\forall r \in \mathcal{R}, \quad F_r = F'_{r}, \quad G_r = G'_{r} \quad \implies \quad [f \succeq g \iff f' \succeq g']$$

Axiom DRE requires that the preference between two acts only depends on the respective position of their consequences (for each state) with respect to reference levels in $\mathcal{R}$. In this respect, it can be seen as the natural variant of the Ordinal Invariance Axiom [4] enforcing ordinal comparisons between acts to rely on reference levels. It can be easily shown that SRL implies DRE, but the converse in not true as shown by the following:

**Example 3** Suppose that $S = \{s_1, s_2, s_3, s_4\}$ and consider $x, y, z$ 3 consequences of $X$ such that $x \succ_X y \succ_X z$. Consider two reference levels $r_1 = x, r_2 = y$ and 4 acts $f, g, f', g' \in A$ such that:

$$
\begin{array}{c|cccc}
  f & x & y & y & z \\
  g & x & y & z & y \\
  f' & y & x & y & z \\
  g' & y & y & z & x \\
\end{array}
\quad
g \succ f 
\quad
g \succ f'
\quad
g \succ f
\quad
g \succ f'

We have $F_1 = G_1 = \{s_1\}, F'_1 = G'_1 = \{s_2\}, F_2 = F'_2 = \{s_1, s_2, s_3\}$ and $G_2 = G'_2 = \{s_1, s_2, s_4\}$. If $f \succ g$ and $g' \succ f'$, then axiom SRL does not hold (just choose $q = 2, i = j = 2, k = 1$ in the formulation of SRL). Yet this situation is compatible with DRE.

In the context of reference-dependent models, the relative likelihood of events can be assessed by observing conflicting events in the respective comparison of consequences of two acts $f$ and $g$ relatively to reference levels. Following this idea, a relative likelihood relation $\succsim_{\Lambda}$ over the set of events $\Lambda = 2^S$ might be derived from $\succsim_X$ by setting, for all $A, B \subseteq S$:

$$A \succsim_{\Lambda} B \iff \exists f, g \in A, \forall r \in \mathcal{R} - \{r\}, F_r = G_r, \quad f \succeq g$$

(4)

Such a construction only makes sense if there is no antagonist pairs of acts $(f, g)$ and $(f', g')$ inducing respectively $A \succsim_{\Lambda} B$ and $B \succsim_{\Lambda} A$ by Equation (4). Axiom SRL excludes such situations: suppose that two subsets of $S$. $A$ and $B$ are such that there exists $f, g \in X$, $f', g' \in X$ such that $f \succeq g$ implies $A \succsim_{\Lambda} B$ and $g' \succ f'$ implies $B \succsim_{\Lambda} A$. If $f \succeq g$ implies $A \succsim_{\Lambda} B$, it means that $\exists r \in \mathcal{R}, F_r = A, \forall p \neq r, F_p = G_p$. If $g' \succ f'$ implies $B \succsim_{\Lambda} A$, it means that $\exists r' \in \mathcal{R}, F'_{r'} = A, G'_{r'} = B$ and $\forall p \neq r, F'_p = G'_p$. Such a situation is not compatible with the respect of axiom SRL.

### 3.2 Ordinal reference dependent rules

Under SRL we can define, for any reference level $r \in \mathcal{R}$ a reference-based preference relation $\succsim_{r}$ over acts by:

$$f \succsim_{r} g \iff F_r \succsim_{\Lambda} G_r$$

(5)

The profile $(\succsim_{r_1}, \ldots, \succsim_{r_q})$ of reference-based preferences relations contains any useful information that might be used in an ordinal reference dependent model. We introduce now an axiom of ordinal invariance with respect to reference-based preferences (OIRP) requiring that preferences between two acts only depend on the relative position of these acts in the $q$ reference-based preferences relations.

**Axiom OIRP** $\forall f, g, f', g' \in A$:

$$\forall r \in \mathcal{R}, \quad f \succsim_{r} g \iff f' \succsim_{r} g' \iff g \succsim_{r} f \iff g' \succsim_{r} f' \quad \implies \quad [f \succsim g \iff f' \succsim g']$$

This axiom is a transposition to reference-based preferences of the Ordinal Equivalence Axiom [4] to characterize Likely Dominance Rules. Without OIRP, nothing guarantees that the preference between two acts $f$ and $g$ only depends on the status of relations $\succsim_{r_j}, j = 1, \ldots, q$ on the pair $(f, g)$. 
Suppose that the DM beliefs about possible events are such that A ≿ₐ B if |A| ≥ |B| (which means that, for example, (x, y, g) ≿ (y, y, y)). Then we have f ≿ₓ g, f ≿ᵧ g, f’ ≿ₓ′ g′, f’ ≿ᵧ′ g′. If DM prefers such that f ≿ g and g’ ≿ f’, which does not contradict axiom SRL, then nothing in the profile (≿ₓ, ≿ᵧ, ≿ₓ, ≿ₓ) can explain the different status of pairs (f, g) and (f’, g’) according to ≿. OIRP precisely avoids such preference situations. Actually it avoids any situation where a preference between two acts might depend on other acts (an idea close to Arrow independence of irrelevant alternative [1]).

Hence SRL and OIRP characterize an interesting class of reference-dependent preferences that can be obtained by ordinal aggregation of the profile (≿ₓ₁, ..., ≿ₓₙ). Among others, one can check that the combination of SRL and OIRP implies P4. We can now state a proposition giving the general form of such reference-dependent relations.

Proposition 1 If the preference relation ≿ on A satisfies axioms SRL and OIRP, then there exists a relation ≿ₐ defined on R = 2ᵗ such that:

\[ f ≿ g \iff \{ r \in R, f ≿ₐ g \} \subseteq \{ r \in R, f ≿ₐ g \} \]

Proof. Thanks to SRL, an important relation ≿ₐ can soundly be revealed from ≿ by Equation (4). We then define a relation ≿ₐ on the subsets of R as follows: consider Q and Q’ two subsets of \( \mathcal{P}(A) \) such that \( Q = \{ r \in R, f ≿ₐ g \} \) and \( Q’ = \{ r \in R, f ≿ₐ g \} \). Hence, suppose that there are two couples of a \( A \times A \) f, g and \( A \times A \) f’ g’ such that \( Q = \{ r \in R, f ≿ₐ g \} \) and \( Q’ = \{ r \in R, f ≿ₐ g \} \). Thanks to axiom OIRP, we have \( f ≿ g \Rightarrow f’ ≿ g’ \), which soundly shows that relation ≿ₐ is soundly defined. Indeed, we cannot have \( Q ≿ₐ Q’ \) with a pair \( f, g \) and \( Q ≿ₐ Q’ \) with another pair \( f’, g’ \).

Relation ≿ₐ reflects the relative importance attached to subsets of consequences by the DM. The reference-based rule given in Equation (6) is formally equivalent to a Likely Dominance Rule [4] applied on reference-dependent relations ≿ₐ, for all \( r \in R \) instead of state-dependent relations ≿ₐ defined by \( f ≿ₐ g \Rightarrow f(s) ≿ₐ X \), for all \( s \in S \). In the next section, we investigate the structure such rules under the assumption of transitivity for ≿ₐ.

4 AXIOMATIC CHARACTERIZATION

The axioms used to establish Proposition 1 are not sufficient to derive a fully operational decision rule. Indeed, nothing guarantees that ≿ₐ constructed according to Equation (6) is transitive or even quasi-transitive. The following example shows indeed that natural relations ≿ₐ based on a majority rule might fail to produce transitive relations, as Condorcet’s triplets can appear in the profile (≿ₓ₁, ..., ≿ₓₙ).

Example 5 Under SLP, we can assume that there exist \( r₁, r₂, r₃ \in R \) such that \( x \succₙ x \succ x \succₙ x \) so that \( x \succ x \succ x \succ x \succ x \succ x \) w. Let \( \mathcal{A} \) be such that \( \mathcal{A} \ni \{ s₁, s₂, s₃ \} \times \{ s₁, s₂, s₃ \} \). Let \( f, g, h \in A \) so that \( f = (x, w, z) \), \( g = (z, x, z) \) and \( h = (y, y, w) \). Then we have \( f \simₐ g \) and \( g \simₐ h \) implies that \( f \simₐ g \).

We can now establish the following representation theorem:

Theorem 1 If the preference relation ≿ on A satisfies P1, WP3, SP5, SRL and IRP then there exists a permutation \( \sigma \) on \( \{ 1, \ldots, q \} \), and a likelihood relation \( \simₐ \) on subsets of \( S \) such that:

\[ f \succ g \iff \sigma(f) \succₐ \sigma(g) \]

Sketch of the proof. Under P1 and SP5, we know there exists at least \( q \geq 3 \) reference levels in \( R = \{ r₁, \ldots, rₙ \} \) so that \( f₁ \succ \ldots \succ fₙ \). Then, the proof proceeds as in the Fishburn’s characterization result for lexicographic preferences [11], translated from a product.
set to Arrow’s original framework [1]. Before, we have to prove that 
profiles \( (\succ_r, i = 1, \ldots, q) \) have the adequate structure. To this end, we first show that \( \succeq \) satisfies the strong Pareto principle, to this 
end, we find \( f > x \), if we consider that \( f \) exceeds \( x \). This is due to the following property:

\[
\forall r \in \mathbb{R} - \{F_r\}, r_i > r_{i+1} \text{ and } r_i > r_{i+1} \text{ by definition of reference levels and } r_i > r_{i+1} \text{ since } S \succ \Lambda \text{ if } P \succ S \text{ (actually SP5)}. So \n\forall f, g \in \mathbb{A}, \forall r \in \mathbb{R}, f \succeq g \text{ and } x \in \mathbb{A}, f \succ x \text{ implies } f > g \text{ and } \text{which proves the strong Pareto principle. Then our IRP axiom plays the role of Fishburn’s non-compensation axiom on product sets. Fi-
nally the richness of the product set structure exploited in Fishburn’s result is obtained here by the diversity of relationships between acts due to the existence of Condorcet triples illustrated by Example 5.} \]

Theorem 1 shows that, under IRP and the transitivity requirement (P1), RLD rules necessarily perform a lexicographic aggregation of relations \( \succ_r, i = 1, \ldots, q \), thus inducing the existence of a ‘dictor’ among reference levels (a transposition of Arrow’s theorem), and even the existence of a hierarchy of reference levels. Reference \( r_i \) is only allowed to discriminate between two acts \( f \) and \( g \) when reference levels \( r_k, k < i \) (higher in the hierarchy) are useless in the comparing \( f \) and \( g \). The hierarchy order is completely character-
ized by permutation \( \sigma \) on \( \{1, \ldots, q\} \) and can easily be revealed by observing preferences of type \( r_i \{r_i\}_{r_{i+1}} > r_j \{r_j\}_{r_{j+1}} \) for any pair \((i, j)\). A very natural example of RLD rule is given by choosing \( \sigma(k) = q + 1 - k \). It amounts to comparing two acts \( f \) and \( g \) by observing their respective total yield to obtain consequences better than a minimal reference level. If this is not sufficient to discriminate them, then a second (more demanding) level is considered and so on...

The key point with such aggregation procedures is that the hier-
archic structure does not apply on states (as is the case for usual LDR rules induced by axiom OI, see [4]) but on consequences. This leaves room for various types of complete monotonic likelihood rel-
ations \( \succeq \), as those induced by capacities (additive or not, decom-
posable or not). In this respect, this offers much more descriptive possibilities than LDR rules. Moreover we have the following nice property which proves compatibility with the qualitative counterpart of stochastic dominance \( \succeq \Delta \) defined on \( \mathbb{A} \) by:

\[
f \succeq \Delta g \iff \forall x \in X, \{s \in S, f(s) \succeq x \} \supseteq \{s \in S, g(s) \succeq x \}
\]

Proposition 3 \( \forall f, g \in \mathbb{A}, f \succ \Delta g \Rightarrow f \succ g \).

Proof. Since \( f \succ \Delta g \) we have: \( \forall r \in \mathbb{R}, F_r \succeq \Lambda G_r \) and \( F_r \succeq \Lambda G_r \), for some \( r \in \mathbb{R} \). Hence the Strong Pareto principle gives \( f \succ g \). \( \square \)

5 CONCLUSION

We have investigated new qualitative models for decision making un-
der uncertainty, characterized by the introduction of reference levels. The main features of the resulting RLD rules are the following:

1) level of information required: models are purely ordinal. The deci-
sion rule is completely characterized by two binary relations \( \succeq \Delta \) (rel-
ative likelihood on events) and \( \succeq X \) (preference over consequences). This deparls from the great majority of models based on a numerical qualitative or quantitative decision criterion, the definition of which relies on the existence of certainty equivalent implicitly mapping util-
ity and uncertainty scales into a single one [4].

2) prescriptive potential: although based on purely ordinal methods, they are fully compatible with the transitivity of preferences (even if a probabilistic likelihood relation is used). The use of reference levels and reference events indeed overcomes the usual limitations of purely ordinal aggregation methods, by moving the application point of Arrow-like theorems. The introduction of SRL instead of OI ex-
changes the role of events and consequences in the ordinal aggrega-
tion. Thus, \( \succeq \) is obtained by aggregation of reference-dependent rel-
ations \( \succeq r, r \in \mathbb{R} \) instead of state-dependent preferences \( \succeq s, s \in S \).

As an important consequence we get Theorem 1 showing that nec-
essary lexicographic structures induced by transitivity emerge on the space of consequences rather than on the space of events (see LDR [3]). In our opinion, this is much more satisfactory.

3) Descriptive potential: many recent studies in decision making un-
der uncertainty concern the violation of the sure-thing principle P2 and the need of models to capture such sophisticated preferences. As Choquet expected utility [18] is a useful extension to EU in the field of quantitative models, as Sugeno Expected Utility [8] is a use-
ful extesion of qualitative utility, RLD rules provide a new family of Freedom Rules allowing, if necessary, violations of P2. As future works, we have to investigate a weaker version of SRL leaving room for a set of \( q \) possibly different reference-dependent likelihood relations. Finally, the connections of our approach with possibilistic lexicographic decision rules proposed in [10] are worth investigating, as well as other characterisation results under quasi-transitivity or without completeness.

REFERENCES

cision theory’, in Proc. 13th National Conference on Artificial Intelli-
preference relations and comparative uncertainty: an axiomatic ap-
chines, 49(4), 455–495, (2002).
[6] D. Dubois, H. Fargier, and H. Prade, ‘Decision-making under ordi-
nal preferences and comparative uncertainty’, in Proc. of UA’97, eds.,
[8] D. Dubois, H. Prade, and R. Sabbadin, ‘Qualitative decision theory with
dice and the elusive principle of indifference’, Scientific American, 223,
[18] D. Schmeidler, ‘Integral representation without additivity’, Proceed-