On Bi-Capacity-based Concordance Rules In Multicriteria Decision Making

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Abstract Several models have been proposed in multi-criteria decision making relying on ordinal information to aggregate the performances on different criteria in order to rank the alternatives, but some situations can still not be well described by these models. We propose here to investigate the interest of new fuzzy measures (bi-capacities) in ordinal aggregation procedures.

Keywords: multicriteria decision making, aggregation methods, fuzzy measure .

Introduction

In the field of multicriteria decision making (MCDM), many models have been proposed to describe preference relations. A first school (see e.g. Keeney and Raiffa [9]) is based on a numerical representation of preferences and put forwards the aggregation of marginal scaled utility functions into a global preference function, using for example the weighted sum or other additive utilities. Some generalizations, using non-additive functions, have been proposed based on the Choquet integral (see e.g. Schmeidler [15], Grabisch [6], Marichal [11]). But in many cases in MCDM, the information available about the alternatives is not sufficient to measure precisely the utility of each alternative on each criterion, or to guaranty the commensurability of the criteria. In such cases, non-numerical approaches based on the use of preferences graphs are useful. In the relational approach of preference models, preferences along each criteria (whether numerical or not) are represented by a binary preference relation, and aggregation methods like votes, majority rules or concordance rules are used to perform criteria aggregation (see e.g. Roy [13], Fodor and Roubens [5], Perny and Roubens [12]). All these methods are based on evaluating the importance of the coalitions of criteria which favor an alternative over another. In this idea, the importance of each criterion, or coalition should be evaluated by a weight, or a fuzzy measure (see Sugeno [16]), or capacity (see Choquet [3]). This has been much studied in the numerical approach, but it has been less studied in preference aggregation (see e.g. the concordance rules proposed by Dubois and Al. [4]). But the use of a capacity to describe the importance of criteria coalitions is sometimes not sufficient. Grabisch and Labreuche [7] recently proposed bi-capacities as a useful generalization of capacities.

The aim of this paper is to investigate the description potential of bi-capacities in graphs-based aggregation methods like concordance rules and to provide representation theorems for bi-capacity-based concordance relations
Section (1) presents the general framework and points out the descriptive limits of some ordinal MCDM models. Section (2) introduces preference models using ordinal bi-capacities and reference points. Section (3) presents the main results : a characterization of preference models using bi-capacities with and without a reference point. All the proofs are relegated in the appendix in order to facilitate the reading.

1 Motivations

1.1 Notations and definitions

Notations
A multicriteria decision problem is characterized by a set X of alternatives and a set $N = \{1, \ldots, n\}$ of attributes used to describe the alter-

natives. We note $\mathcal{P}(N)$ the set of subsets of N. Let $(X_i, \succsim_i)$ denote a finite ordered scale where X_i is the set of attribute values for component $i \in N$, and $\succsim_i$ is a complete weak-order on X_i. $X = X_1 \times \ldots \times X_n$ is said to be the multicriteria space. We suppose that each attribute set X_i is composed by at least 3 different levels.
We note $C_\succ(x, y)$ the set $\{j \in N, \ x_j \succ_j y_j\}$.

An aggregation procedure consists in deriving a global preference relation $\succsim$ on X from partial preferences relations $\succsim_j$ on each criterion $X_1, \ldots, X_n$. $\succsim$ is supposed to be a weak-order, i.e. a transitive and reflexive binary relation.

Many models have been proposed to describe aggregation procedures in the frame of ordinal multicriteria decision making, see for example Roy [14] for a review.

Generalized Concordance Rules
Dubois and al. [4] have introduced the Generalized Concordance Rules in a purely ordinal frame as follow:

Definition 1 *A generalized concordance rule defines a preference relation $\succsim$ on X from the relations $\succsim_j$ on X_j, $\forall j = 1, .., n$ as follows :*

$$x \succsim y \iff C_\succ(x, y) \succsim_N C_\succ(y, x)$$

where $\succsim_N$ is a relative importance relation on $\mathcal{P}(N)$

Capacity-based Concordance Rules
Let us recall the definition of fuzzy measures, or capacities for finite sets. For more details, see e.g. Sugeno and Murofushi [17], Grabisch and Roubens [8] and Grabisch and Labreuche [7].

Definition 2 *A capacity $\mu : 2^N \to \mathbb{R}^+$ is a set function such that $\mu(\emptyset) = 0$, and $A \subseteq B \subseteq N$ implies $\mu(A) \leq \mu(B)$. The capacity is normalized if in addition $\mu(N) = 1$.*

An instance of the general model introduced in definition 1 is:

Definition 3 *A capacity-based concordance rule defines a preference relation $\succsim$ on X from the relations $\succsim_j$ on X_j, $\forall j = 1, .., n$ and a capacity μ on $\mathcal{P}(N)$ as follows :*

$$x \succsim y \iff \mu(C_\succ(x, y)) \geq \mu(C_\succ(y, x))$$

1.2 Some limits of existing models

Even if the Capacity-Based Concordance Rules and the Generalized Concordance Rules can describe many preference relations, the following examples show different situations where Capacity-Based Concordance Rules, or even Generalized Concordance Rules are unable to explain the proposed preferences.

Example 1 *Let us consider the following example, giving evaluations obtained by several cars following three criteria : Comfort (1), Price (2) and Consumption (3). The performances on each criteria are evaluated on three different scales : three levels "+", "=" and "-" for the comfort, five categories from A (the cheapest) to E (the more expensive) for he price, and the consummation in liter for hundred kilometers.*

	1	*2*	*3*
x^1	+	*B*	*8*
x^2	+	*E*	*6.5*
x^3	-	*B*	*6.5*

Assume that the decision maker has the following preferences : she prefers x^1 to x^3 (so $x^1 \succ x^3$), but she has no preferences between x^1 and x^2, and x^2 and x^3 (so $x^1 \sim x^2$ and $x^2 \sim x^3$).

The question is : can we represent these preferences with a Capacity-Based Concordance Rule? In this model, we have $x \succsim y$ if and only if $\mu(C_\succ(x, y)) \geq \mu(C_\succ(y, x))$. We can easily see that in the present situation, $x^1 \sim x^2$ implies that $\mu(\{2\}) = \mu(\{3\})$, $x^2 \sim x^3$ implies that $\mu(\{1\}) = \mu(\{2\})$ while $x^1 \succ x^3$ implies that $\mu(\{1\}) > \mu(\{3\})$: there is an impossibility.

Remark : We have shown in this example that some preference relations cannot be represented by a Capacity-Based Concordance Rule. But these preferences can be represented by a Generalized Concordance Rule with a non transitive importance set relation $\succsim_N$ specifying $\{M\} \sim_N \{P\}$, $\{P\} \sim_N \{C\}$ and $\{M\} \succ_N \{C\}$. These relations can also be represented by a Bi-Capacity-Based Concordance relation, as seen below.

Example 2 *Let us consider now the following example in the same frame than example (1), includ-*

ing a fourth criterion representing the available options (air-bags, high fidelity radio...) evaluating by 1 (few options), 2 or 3 (many options).

	1	2	3	4
x^1	+	C	8	2
x^2	+	C	6	1
x^3	=	A	8	2
x^4	=	A	6	1

Assume that the decision maker has the following preferences : he prefers x^1 to x^2 (so $x^1 \succ x^2$), and x^4 to x^3 (so $x^4 \succ x^3$).

The question is : can we represent preferences $x^1 \succ x^2$ and $x^4 \succ x^3$ using a Capacity-Based Concordance rule or even a Generalized Concordance Rule defined in definition 1 ? We can easily see that in the present situation, $x^1 \succ x^2$ implies that $\{3\}$ is more important than $\{4\}$, while $x^4 \succ x^3$ implies the contrary : there is an impossibility. The Concordance rules introduced above are not sufficient to be able to describe some existing preferences.

2 Bi-capacity-based concordance models

We propose in this section preference models using a bi-capacity to represent the issue of conflicting coalitions in concordance rules.

We will first recall the definition of bi-capacities. Let us denote $\mathcal{Q}(N) = \{(A, B) \in \mathcal{P}(N) \times \mathcal{P}(N) \mid A \cap B = \emptyset\}$, where $\mathcal{P}(N)$ stands for 2^N as introduced in Grabisch and Labreuche [7].

Definition 4 *A function $\nu : \mathcal{Q}(N) \to \mathbb{R}$ is a bi-capacity if it satisfies :*

1. $\nu(\emptyset, \emptyset) = 0$
2. $A \subseteq B$ *implies* $\nu(A, .) \leq \nu(B, .)$ *and* $\nu(., A) \geq \nu(., B)$

In addition, ν is normalized if $\nu(N, \emptyset) = 1 = -\nu(\emptyset, N)$.

We can then define bi-capacity-based concordance rules as follows :

Definition 5 *A bi-capacity-based concordance rule defines a preference relation $\succsim$ on X from the relations $\succsim_j$ on X_j, $\forall j = 1, .., n$ and a bi-capacity ν on $\mathcal{P}(N) \times \mathcal{P}(N)$ as follows :*

$$x \succsim y \iff \nu(C_{\succ}(x, y), C_{\succ}(y, x)) \geq 0$$

Return to Example 1 : *The preference relations described in Example 1 can easily be built using a Bi-Capacity-Based Concordance rule as follows :*

$$\begin{aligned} \nu(\{2\}, \{3\}) &= 0 \\ \nu(\{1\}, \{2\}) &= 0 \\ \nu(\{1\}, \{3\}) &= 1 \end{aligned}$$

Note that the use of the bi-capacity-based concordance relations is interesting only if $C_{\succ}(x, y) \neq N \backslash C_{\succ}(y, x)$. If not, the bi-capacity reduces to a capacity μ by setting $\mu(A) = (\nu(A, \overline{A}) + 1)/2$.

Return to Example 2 : *The preference relations described in Example 2 can not be built using a Bi-Capacity-Based Concordance rule, as seen below : $C_{\succ}(x^1, x^2) = C_{\succ}(x^3, x^4) = \{4\}$, $C_{\succ}(x^2, x^1) = C_{\succ}(x^4, x^3) = \{3\}$, so $x^1 \succ x^2$ implies that $\nu(\{4\}, \{3\}) > 0$ and $x^4 \succ x^3$ implies that $\nu(\{3\}, \{4\}) > 0$, which is not coherent.*

Introducing a reference point in a bi-capacity-based concordance rule means that the preference relation between two elements x and y will no longer depend on the position of x relatively to y. Only the respective position of each alternative compared to a reference point p are taken into account to compare x and y. We will note $p = (p_1, \dots, p_n) \in X$ the *reference point*, and assume that
- $\forall j \in N, \ \exists x, y \in X$ such that $x_j \succ_j p_j \succ_j y_j$ (p is neither majoring nor minoring any criteria).
- $\forall j \in N, \ \forall x \in X, \ x_j \succsim_j p_j$ or $p_j \succsim_j x_j$ (each element of X_j is comparable to p_j for $j = 1, \dots, n$).

We can define a bi-capacity-based concordance rule with a reference point as follow :

Definition 6 *A bi-capacity-based concordance rule with a reference point defines a preference relation $\succsim$ on X from the relations $\succsim_j$ on X_j, $\forall j = 1, .., n$, a bi-capacity ν on $\mathcal{P}(N) \times \mathcal{P}(N)$ and a reference point p as follows : $x \succsim y \iff \nu(C_{\succ}(x, p), C_{\succ}(p, x)) \geq \nu(C_{\succ}(y, p), C_{\succ}(p, y))$*

Return to example 2 : *Let us take p=(=,C,6,2) as a reference profile for a car. Then* $C_\succ(x^1,p) = \{1\}$, $C_\succ(p,x^1) = \{3\}$, $C_\succ(x^2,p) = \{1\}$, $C_\succ(p,x^2) = \{4\}$, $C_\succ(x^3,p) = \{2\}$, $C_\succ(p,x^3) = \{3\}$, $C_\succ(x^4,p) = \{2\}$, $C_\succ(p,x^4) = \{4\}$. *Hence, preferences* $x^1 \succ x^2$ *and* $x^4 \succ x^3$ *are easily represented by a Bi-Capacity-Based Concordance rule with* $\nu(\{1\},\{3\}) > \nu(\{1\},\{4\})$ *and* $\nu(\{2\},\{4\}) > \nu(\{2\},\{3\})$.

We have seen above that the introduction of a reference point allows the decision maker to sort the value of each alternative on each criterion in two categories : good (better than p) or bad (worse than p). In order to be a little bit more discriminating, we can introduce a second reference point, with for example $p_i^1 \succ_i p_i^2 \ \forall \in N$, which consists, for each criteria, in partitioning the different criterion values in three categories : good (better than p_i^1), medium (between p_i^1 and p_i^2) and bad (worse than p_i^2). Several approaches can then be considered to compare the alternatives to p^1 and p^2. We choose here an up to down filtering, where the alternatives are first compared to p^1 and then to p^2, in order to select first the best alternatives, but other choices are also possible. We will so define a bi-capacity-based concordance rule with two reference points as follow :

Definition 7 *A bi-capacity-based concordance rule with two reference points defines a preference relation* $\succsim$ *on* X *from the relations* $\succsim_j$ *on* X_j, $\forall j = 1,..,n$, *a bi-capacity* ν *on* $\mathcal{P}(N) \times \mathcal{P}(N)$ *and two reference points* p^1*and* p^2 *as follows :* $x \succsim y \iff \nu(C_\succ(x,p^1), C_\succ(p^2,x)) \geq \nu(C_\succ(y,p^1), C_\succ(p^2,y))$

Example 3 *In this example, we show the interest of the introduction of a second reference point. The context is the same as in example 2.*

	1	*2*	*3*	*4*
x^1	+	*A*	*6*	*2*
x^2	+	*C*	*6*	*2*
x^3	+	*E*	*6*	*2*
x^4	-	*C*	*9*	*3*

Assume that the decision maker has the following preferences : she prefers $x^1 \succ x^2$ *and* $x^2 \succ x^3$. *Assume also that she prefers* $x^2 \succ x^4$, *which shows that preference* $\succsim$ *does not depend only on criterion (2).*

The question is : can we represent these preferences with a bi-capacity-based concordance rule with only one reference point p? Suppose that it is possible, and let us note $p = (p_1, p_2, p_3, p_4)$ the reference point. We show that in these conditions, there is not acceptable value for one of the p_i, value of p on criterion i. First, the fact that $x^1 \succ x^2$ gives, on criterion 2, that $A \succ_2 p_2 \succsim_2 C$, because it is the only mean to distinguish x^1 and x^2. Then, the fact that $x^2 \succ x^3$ gives, on the same criterion, that $C \succ_2 p_2 \succsim_2 E$. so we have $p_2 \succsim_2 C \succ p_2$ which is not possible. We cannot represent these preferences with this model.

But these preferences can easily be build with a bi-capacity-based concordance rule with two reference points p^1 and p^2 as follow : let us take $p^1 = (=, B, 7, 2)$ and p^2=(=,D,9,2). Then, following the definition (7), we have

$$x^1 \succ x^2 \Rightarrow \nu(\{1,2,3\},\emptyset) > \nu(\{1,3\},\emptyset)$$
$$x^2 \succ x^3 \Rightarrow \nu(\{1,3\},\emptyset) > \nu(\{1,3\},\{2\}).$$

3 Axiomatization

Establishing a representation theorem for a specific decision model consists in giving a set of conditions on the preference relation $\succsim$, through some testable axioms, and proving that they are necessary and sufficient for $\succsim$ to be represented by the proposed model. A representation theorem of a model allows to justify theoretically the use of this model in a specific context. Several general representation theorems on product sets have been proposed in conjoint measurement theory (see e.g. Luce *et al.* [10], Bouyssou and Pirlot [2]). Specific representation theorems for concordance rules have been proposed by Bouyssou and Pirlot [1] and Dubois *et al.* [4]. In this section, we establish three representation theorems to characterize preference structures which are compatible with a bi-capacity-based concordance rule as expressed by the relations defined in definition 5, 6 and 7.

3.1 Bi-capacity-based concordance model

First of all, we need to specify that the preference structure is ordinal-based. This is what the following axiom, introduced in Dubois *et al.* [4], says.

Axiom 1 Neutrality and Independence (NI) $\forall x, y, z \in X$, $[C_{\succ}(x, y) = C_{\succ}(z, w)$ *and* $C_{\succ}(y, x) = C_{\succ}(w, z)] \Rightarrow [x \succsim y \iff z \succsim w]$

Then, it seems reasonable for the preference structure to have also a unanimity property on the criteria.

Axiom 2 Unanimity (UNA)

$$\forall i = 1, \ldots, n,\ x_i \succsim_i y_i \Rightarrow x \succsim y$$

These two axioms are enough to specify which kind of preference structures can be characterized by a bi-capacity-based concordance rule, as shown in the following theorem:

Theorem 1 *If the preference relation $\succsim$ and the weak-orders $\succsim_j$ satisfy axioms (NI) and (UNA), then a bi-capacity ν exists such that :*

$$x \succsim y \iff \nu(C_{\succ}(x, y), C_{\succ}(y, x)) \geq 0$$

3.2 Bi-capacity-based concordance model with a reference point

Introducing a reference point in a bi-capacity-based concordance rule means that the NI axiom is no longer respected. It is now important to notice that only the respective position of each alternative compared to the reference point p are taken into account to compare two different alternatives. This is the meaning the following axiom:

Axiom 3 Neutrality and Independence with respect to a Reference Point (NIp)
$\forall x\ , y \in X\ , \left\{ \begin{array}{l} C_{\succ}(x, p) = C_{\succ}(y, p) \\ C_{\succ}(p, x) = C_{\succ}(p, y) \end{array} \right. \Rightarrow x \sim y$

Associated with the unanimity axiom, (NIp) is sufficient to characterize the preference relations which can be represented by a bi-capacity-based concordance relation, as shown in the following theorem:

Theorem 2 *If $\succsim$ and $\succsim_j$ satisfies axioms (NIp) and (UNA), then a bi-capacity ν exists such that : $x \succsim y \iff \nu(C_{\succ}(x, p), C_{\succ}(p, x)) \geq \nu(C_{\succ}(y, p), C_{\succ}(p, y))$*

Remark : as we have seen before in Example 2, bi-capacity-based concordance rule using a reference point allows us to capture by ordinal concordance rules preference relations which were not modelled before.

3.3 Bi-capacity-based concordance model with two reference points

The neutrality and independence axiom should take into account the existence of two different reference points as follow :

Axiom 4 Neutrality and Independence with respect to two Reference Points (NI2p)
$\forall x, y \in X$,
$\left\{ \begin{array}{l} C_{\succ}(x, p^1) = C_{\succ}(y, p^1) \\ C_{\succ}(p^2, x) = C_{\succ}(p^2, y) \end{array} \right. \Rightarrow x \sim y$

As above, this axiom is sufficient, associated with an unanimity axiom, to characterize the bi-capacity-based concordance rules with two reference points, as shown in the following theorem:

Theorem 3 *If $\succsim$ and $\succsim_j$ satisfies axioms (NI2p) and (UNA), then a bi-capacity ν exists such that: $x \succsim y \iff \nu(C_{\succ}(x, p^1), C_{\succ}(p^2, x)) \geq \nu(C_{\succ}(y, p^1), C_{\succ}(p^2, y))$*

4 Conclusion

New models using bi-capacity based concordance rules have been proposed to enlarge the description capacity of Concordance rules in ordinal MCDM. For examples, in the situations where the preference relations are semi-transitives, bi-capacity based concordance relations cannot be reduced to capacity based concordance relations.

On another hand, the introduction of reference points in such models, allows us to get out the frame of the so-called Arrow's impossibility theorem, and so to obtain transitive and non dictatorial preference rules based on the aggregation of ordinal preferences.

Appendix

We define $z = (x_j, y_{-j})$ as the element of X such as $z_j = x_j$ and $z_i = y_i \;\; i \neq j$, with $j \in N$ and $x, y \in X$. More generally, $z = (x_A, y_{-A})$ is defined as the element of X such as $z_j = x_j$ if $j \in A$ and $z_j = y_j$ if $j \notin A$, with $A \subset N$ and $x, y \in X$.

Proof of theorem 1

Let us define a function f from $\mathcal{P}(N) \times \mathcal{P}(N)$ to $\{-1, 0, 1\}$ by :
$f(A,B) = 1 \iff \exists x, y \in X | C_\succ(x,y) = A,\; C_\succ(y,x) = B$ and $x \succ y$
$f(A,B) = 0 \iff \exists x, y \in X | C_\succ(x,y) = A,\; C_\succ(y,x) = B$ and $x \sim y$
$f(A,B) = -1 \iff \exists x, y \in X | C_\succ(x,y) = A,\; C_\succ(y,x) = B$ and $y \succ x$

Let us show that the function f is well defined : suppose that $\exists$ (x,y) and $(z,w) \in X \times X$ such as $C_\succ(x,y) = C_\succ(z,w) = A$ and $C_\succ(y,x) = C_\succ(w,z) = B$. As $\succsim$ respects the axiom (NI), we are sure that $x \succsim y \iff z \succsim w$, and so there is no ambiguity on the value of $f(A,B)$. We now show that f is a bi-capacity.

- $f((\emptyset, \emptyset)) = 0$: if $C_\succ(x,y) = C_\succ(y,x) = \emptyset$, it means that $\forall i \in N$, $x_i = y_i$, and so $x = y$ an d$x \sim y$, which means that $f((\emptyset, \emptyset)) = 0$.
- Monotonicity : suppose that $\exists x, y, z \in X$ such as $C_\succ(x,y) = A$, $C_\succ(z,y) = A'$, $C_\succ(y,x) = C_\succ(y,z) = B$ and $A \subseteq A'$. Do we have $f(A',B) \geq f(A,B)$? Let us take $w \in X$ such as $\forall i \in N$, $w_i = max_{\succsim_i}\{x_i, z_i\}$. We have $\forall i \in N$, $w_i \succsim_i x_i$, so as $\succsim$ respects (UNA), $w \succsim x$. Moreover, as $A \subseteq A'$, $C_\succ(w,y) = A'$ and $C_\succ(y,w) = B$. So $w \succsim y \iff z \succsim y$.
  - if $f(A,B) = -1$, then $f(A',B) \geq -1$ and so $f(A',B) \geq f(A,B)$
  - if $f(A,B) = 0$, we have $x \sim y$, and, by transitivity, $w \succsim y$. By (NI), we have $z \succsim y$ and so $f(A',B) \geq 0 = f(A,B)$.
  - if $f(A,B) = 1$, we have $x \succ y$, and, by transitivity, $w \succ y$. By (NI), we have $z \succ y$ and so $f(A',B) = 1 \geq f(A,B)$.

  So if $A \subseteq A'$, we always have $f(A,.) \leq f(A',.)$. We can show on the same idea that if $A \subseteq A'$, we always have $f(.,A) \geq f(.,A')$.

This proves that f is a bi-capacity. $\square$

4.1 Proof of theorem 2

Let us define a relation $\succsim'$ on $\mathcal{Q} \times \mathcal{Q}$ by $(A,B) \succsim' (C,D) \iff \exists x, y \in X | \begin{cases} A = C_\succ(x,p) \\ B = C_\succ(p,x) \\ C = C_\succ(y,p) \\ D = C_\succ(p,y) \end{cases}$ and $x \succsim y$.

We show that this relation is a complete weak order on $\mathcal{Q}$.

Let us demonstrate first that the relation $\succsim'$ defined above exists. The relation $\succsim'$ defined above should not depend on the elements x and y chosen during the construction. For this, we should show that if two couples of $X \times X$, (x,y) and (z,w), exist such that $\begin{cases} A = C_\succ(x,p) = C_\succ(z,p) \\ B = C_\succ(p,x) = C_\succ(p,z) \end{cases}$ and $\begin{cases} C = C_\succ(y,p) = C_\succ(w,p) \\ D = C_\succ(p,y) = C_\succ(p,w) \end{cases}$, then $x \succsim y \iff z \succsim w$. This is obvious, following (NIp), because $\begin{cases} C_\succ(x,p) = C_\succ(z,p) \\ C_\succ(p,x) = C_\succ(p,z) \end{cases}$ and $\begin{cases} C_\succ(y,p) = C_\succ(w,p) \\ C_\succ(p,y) = C_\succ(p,w) \end{cases}$ give $x \sim z$ and $y \sim w$.

Let us now demonstrate that $\succsim'$ is a weak-order on $\mathcal{Q}$.

- **asymmetry of $\succ'$:** if $(A,B) \succ' (C,D)$, this means that exist $x, y \in X$ such that $\begin{cases} A = C_\succ(x,p) \\ B = C_\succ(p,x) \end{cases}$, $\begin{cases} C = C_\succ(y,p) \\ D = C_\succ(p,y) \end{cases}$ and $x \succ y$. If $(A,B) \prec' (C,D)$, this means that exist $z, w \in X$ such that $\begin{cases} A = C_\succ(z,p) \\ B = C_\succ(p,z) \end{cases}$, $\begin{cases} C = C_\succ(w,p) \\ D = C_\succ(p,w) \end{cases}$ and $w \succ z$. Following (NIp), we have $x \sim z$ and $y \sim w$, which is in contradiction with $x \succ y$ and $w \succ z$. So we can 't have on the same time $(A,B) \succ' (C,D)$ and $(A,B) \prec' (C,D)$, which proves the asymmetry of $\succ'$.

- **symmetry of $\sim'$:** on the same idea, symmetry $\sim'$ is easily shown based on the symmetry of $\sim$ and (NIp).

- **transitivity of $\succsim'$:** if $(A,B) \succsim' (C,D)$ and $(C,D) \succsim' (E,F)$, then exist $x, y, y', z \in X$ such that $\left\{\begin{array}{l} A = C_\succ(x,p) \\ B = C_\succ(p,x)) \end{array}\right.$, $\left\{\begin{array}{l} C = C_\succ(y,p) \\ D = C_\succ(p,y) \end{array}\right.$ and $x \succsim y$, and $\left\{\begin{array}{l} C = C_\succ(y',p) \\ D = C_\succ(p,y') \end{array}\right.$, $\left\{\begin{array}{l} E = C_\succ(z,p) \\ F = C_\succ(p,z) \end{array}\right.$ and $y' \succsim z$. Axiom (NIp) implies that $y \sim y'$, and transitivity of $\succsim$ implies $x \succsim z$, which shows the transitivity of $\succsim'$.

We now show the completeness of $\succsim'$. As we have $\forall i \in N, \exists\ x_i, y_i$ such that $x_i \succ_i p_i \succ_i y_i$, we can build for all $(A,B) \in \mathcal{Q}$ a element $z \in X$ such that $C_\succ(z,p) = A$ and $C_\succ(p,z) = B$: we just have to take $\left\{\begin{array}{l} z_i = a_i \text{ if } i \in A \\ z_i = b_i \text{ if } i \in B \\ z_i = p_i \text{ if not} \end{array}\right.$, with $a_i \succ_i p_i \forall i \in N$ and $b_i \prec_i p_i \forall i \in N$. Completeness of $\succsim$ gives the completeness of $\succsim'$.

As $\succsim'$ is complete and $\mathcal{Q}$ is finite, there is a function $\nu : \mathcal{Q} \to \mathbb{R}$ such that $(A,B) \succsim' (C,D) \iff \nu(A,B) \geq \nu(C,D)$. We now show that this set-function ν is a bi-capacity. Suppose that $A \subseteq B$. Let compare $\nu(A,C)$ and $\nu(B,C)$, with $A \cap C = B \cap C = \emptyset$. Suppose $x \in X$ such that $C_\succ(x,p) = A$ and $C_\succ(p,x) = C$. Suppose $y \in X$ such that $C_\succ(y,p) = B$ and $C_\succ(p,y) = C$. Let denote $z = (x_{A \cup N \backslash B}, y_{B \backslash A})$. Following axiom (UNA), $z \succsim x$, because $\forall j \in N,\ \ z_j \succsim_j x_j$. Actually, if $j \in B \backslash A$, we have $z_j = y_j \succ_j p_j$ and $x_j = p_j$. So $z \succsim x$ gives $(B,C) \succsim' (A,C)$, which implies $\nu(B,C) \geq \nu(A,C)$. We have shown that $A \subseteq B \Rightarrow \nu(A,\cdot) \leq \nu(B,\cdot)$. On the same way, we can show that $A \subseteq B \Rightarrow \nu(\cdot,A) \geq \nu(B,\cdot)$. Then, ν is a bi-capacity if $\nu(\emptyset,\emptyset) = 0$. If not, we just have to take $\nu' = \nu - \nu(\emptyset,\emptyset)$ to have a bi-capacity. □

Proof of theorem 3

let us define a relation $\succsim'$ on $\mathcal{Q} \times \mathcal{Q}$ by $(A,B) \succsim' (C,D) \iff \exists x, y \in X| \left\{\begin{array}{l} A = C_\succ(x,p^1) \\ B = C_\succ(p^2,x) \\ C = C_\succ(y,p^1) \\ D = C_\succ(p^2,y) \end{array}\right.$ and $x \succsim y$.

We show that this relation is a complete weak order on $\mathcal{Q}$.

Let us demonstrate first that the relation $\succsim'$ defined above exists. The relation $\succsim'$ defined above should not depend on the elements x and y chosen during the construction. For this, we should show that if two couples of $X \times X$ (x,y) et (z,w) such that $\left\{\begin{array}{l} A = C_\succ(x,p^1) = C_\succ(z,p^1) \\ B = C_\succ(p^2,x) = C_\succ(p^2,z) \end{array}\right.$ and $\left\{\begin{array}{l} C = C_\succ(y,p^1) = C_\succ(w,p^1) \\ D = C_\succ(p^2,y) = C_\succ(p^2,w) \end{array}\right.$, then $x \succsim y \iff z \succsim w$. It is obvious following (NI2p) : $\left\{\begin{array}{l} C_\succ(x,p^1) = C_\succ(z,p^1) \\ C_\succ(p^2,x) = C_\succ(p^2,z) \end{array}\right.$ and $\left\{\begin{array}{l} C_\succ(y,p^1) = C_\succ(w,p^1) \\ C_\succ(p^2,y) = C_\succ(p^2,w) \end{array}\right.$ gives $x \sim z$ and $y \sim w$.

We now demonstrate that $\succsim'$ is a weak order on $\mathcal{Q}$.

- **asymmetry of $\succ'$:** if $(A,B) \succ' (C,D)$, this means that exist $x, y \in X$ such that $\left\{\begin{array}{l} A = C_\succ(x,p^1) \\ B = C_\succ(p^2,x)) \end{array}\right.$, $\left\{\begin{array}{l} C = C_\succ(y,p^1) \\ D = C_\succ(p^2,y) \end{array}\right.$ and $x \succ y$. If $(A,B) \prec' (C,D)$, this means that exist $z, w \in X$ such that $\left\{\begin{array}{l} A = C_\succ(z,p^1) \\ B = C_\succ(p^2,z)) \end{array}\right.$, $\left\{\begin{array}{l} C = C_\succ(w,p^1) \\ D = C_\succ(p^2,w) \end{array}\right.$ and $w \succ z$. Following the axiom (NI2p), we have $x \sim z$ and $y \sim w$ which is in contradiction with $x \succ y$ and $w \succ z$. So we can 't have on the same time $(A,B) \succ' (C,D)$ and $(A,B) \prec' (C,D)$, which proves the asymmetry of $\succ'$.

- **symmetry of $\sim'$:** on the same idea, symmetry $\sim'$ is easily shown based on the symmetry of $\sim$ and (NIp).

- **transitivity of $\succsim'$:** If $(A,B) \succsim' (C,D)$ and $(C,D) \succsim' (E,F)$, then exist $x, y, y', z \in X$ such that $\left\{\begin{array}{l} A = C_\succ(x,p^1) \\ B = C_\succ(p^2,x)) \end{array}\right.$, $\left\{\begin{array}{l} C = C_\succ(y,p^1) \\ D = C_\succ(p^2,y) \end{array}\right.$ and $x \succsim y$, and

$$\begin{cases} C = C_\succ(y', p^1) \\ D = C_\succ(p^2, y') \end{cases}, \quad \begin{cases} E = C_\succ(z, p^1) \\ F = C_\succ(p^2, z) \end{cases}$$

and $y' \succsim z$. Axiom (NI2p) implies that $y \sim y'$, and transitivity of $\succsim$ implies $x \succsim z$, which proves the transitivity of $\succsim'$.

We now demonstrate the completeness of $\succsim'$. As we supposed that $\forall i \in N$, $\exists \ x_i, y_i$ such that $x_i \succ_i p_i^1$ and $p_i^2 \succ_i y_i$, we can build for all $(A, B) \in \mathcal{Q}$ an element $z \in X$ such that $C_\succ(z, p^1) = A$ and $C_\succ(p^2, z) = B$: we just have to take $\begin{cases} z_i = a_i \text{ if } i \in A \\ z_i = b_i \text{ if } i \in B \\ z_i = c_i \text{ if not} \end{cases}$, with $a_i \succ_i p_i^1$ $\forall i \in N$, $b_i \prec_i p_i^2$ $\forall i \in N$ and $p_i^1 \succsim_i c_i \succsim_i p_i^2$ $\forall i \in N$. Completeness of $\succsim$ implies the completeness of $\succsim'$. As $\succsim'$ is complete and $\mathcal{Q}$ is finite, there is a function $\nu : \mathcal{Q} \to \mathbb{R}$ such that $(A, B) \succsim' (C, D) \iff \nu(A, B) \geq \nu(C, D)$. We now show that this function ν is a bi-capacity. Suppose that $A \subseteq B$. Let compare $\nu(A, C)$ and $\nu(B, C)$, with $A \cap C = B \cap C = \emptyset$. Let take $x \in X$ such that $C_\succ(x, p^1) = A$ and $C_\succ(p^2, x) = C$. Let take $y \in X$ such that $C_\succ(y, p^1) = B$ and $C_\succ(p^2, y) = C$. We note $z = (x_{A \cup N \setminus B}, y_{B \setminus A})$. Following axiom (UNA), $z \succsim x$, because $\forall j \in N$, $z_j \succsim_j x_j$. Actually, if $j \in B \setminus A$, we have $z_j = y_j \succ_j p_j^1$ and $p_j^1 \succsim_j x_j \succsim_j p_j^2$. $z \succsim x$ gives $(B, C) \succsim' (A, C)$, which gives $\nu(B, C) \geq \nu(A, C)$. we have shown that $A \subseteq B \Rightarrow \nu(A, \cdot) \leq \nu(B, \cdot)$. On the same way, we can show that $A \subseteq B \Rightarrow \nu(\cdot, A) \geq \nu(B, \cdot)$. Then, ν is a bi-capacity if $\nu(\emptyset, \emptyset) = 0$. If not, we just have to take $\nu' = \nu - \nu(\emptyset, \emptyset)$ to have a bi-capacity. □

References

[1] D. Bouyssou and J.C. Pirlot. A characterization of strict concordance relations. In *Aiding decisions with multicriteria*, pages 121–146. Kluwer's academic publishers, 2002.

[2] D. Bouyssou and M. Pirlot. Non transitive decomposable conjoint measurement. *Journal of Mathematical Psychology*, 46:677–703, 2002.

[3] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, (5):131–295, 1953.

[4] D. Dubois, H. Fargier, P. Perny, and H. Prade. A characterization of generalized concordance rules in multicriteria decision making. *International Journal of Intelligent Systems, special issue on preference modeling application*, (18):751–774, 2003.

[5] J. Fodor and M. Roubens. *Fuzzy preference modelling and multicriteria decision support*. Kluwer Academic Publishers, 1994.

[6] M. Grabisch. The application of fuzzy integrals in multicriteria decision making. *European Journal of Operational Research*, 89:445–456, 1996.

[7] M. Grabisch and C. Labreuche. Bi-capacities. In *1st Joint International Conference on Soft Computing and Intelligent Systems and 3rd International Symposium on Advanced Intelligent Systems*, 2002.

[8] M. Grabisch and M. Roubens. Application of the Choquet integral in multicriteria decision making. In M. Grabisch, T. Murofushi, and M. Sugeno, editors, *Fuzzy Measures and Integrals - Theory and Applications*, pages 348–374. Physica Verlag, 2000.

[9] R.L. Keeney and H. Raiffa. *Decisions with multiple objectives: Preferences and value tradeoffs*. J. Wiley, New York, 1976.

[10] R.D. Luce, D.H. Krantz, P. Suppes, and A. Tversky. *Foundations of measurement*, volume 3: Representation, axiomatisation and invariance. Academic Press, New York, 1990.

[11] J.L. Marichal. An axiomatic approach of the discrete Choquet integral as a tool to aggregate interacting criteria. *IEEE Transactions on Fuzzy Systems*, 8(6):800–807, December 2000.

[12] P. Perny and M. Roubens. *Preference Modelling*, pages 69–101. Kluwer Academic Publishers, 1998.

[13] B. Roy. How outranking relations helps multiple criteria decision making. In J. Cochrane and M. Zeleny, editors, *Multicriteria Decision Making*, pages 179–201. University of Carolina, 1973.

[14] B. Roy. *Multicriteria Methodology for Decision Aiding*. Kluwer Academic, Dordrecht, 1996.

[15] D. Schmeidler. Integral representation without additivity. *Proc. Amer. Math. Soc.*, (97):255–261, 1986.

[16] M. Sugeno. *Theory of fuzzy integrals and its applications*. PhD thesis, Tokyo Institute of Technology, 1974.

[17] M. Sugeno and T. Murofushi. Choquet integral as an integral form for a general class of fuzzy measures. In *2nd IFSA Congress*, pages 408–411, 1987.