Reference-based Preferences Aggregation Procedures in Multicriteria Decision Making

Antoine Rolland

Laboratoire ERIC - Université Lumière Lyon 2
5 avenue Pierre Mendes-France
F-69676 BRON Cedex - FRANCE
antoine.rolland@univ-lyon2.fr

Abstract

This paper aims at introducing and investigating a new family of merely qualitative models for multicriteria decision making. Such models do not require any numerical representation. Within this family, we will focus on decision rules using reference levels in order to help comparing several alternatives. We will investigate both the descriptive potential of such rules and their axiomatic foundations. After recalling the descriptive and prescriptive limitations of merely ordinal rules that do not use reference points, we will introduce a new axiom requiring that the Decision Maker's preference between two alternatives depends on the respective positions of their consequences relatively to reference levels. Under this assumption we will determine the only possible form for the decision rule and characterize some particular instances of this rule under transitivity constraints. Our results show that introducing reference points overcomes the usual limitations of purely ordinal aggregation methods, by moving the application point of Arrow's theorem.

Keywords: multi-criteria decision making; aggregation of preference relations; reference levels

1. Introduction

A classical problem in multicriteria decision making (MCDM) is about aggregating preference relations on each criterion to obtain a global preference relation on the set of the considered alternatives. Several axiomatic papers show the theoretical and practical difficulties in aggregating preference relations which are partially in conflict. Impossibility theorems, such as those proposed by Arrow (1951) or Gibbard (1973) and Satterthwaite (1975), explain why it is difficult to build a good MCDM method. But most of these results are based on the independence of irrelevant alternatives hypothesis. Such an hypothesis allows only to use direct comparison of two alternatives to determine their respective preferences. Another interesting option exists: we can compare two alternatives by examining their respective levels compared to some reference points (e.g. specific profiles, exigence levels, typical cases). These kinds of decision rules are already used in sorting problems in MCDM (see e.g. Figueira et al. (2005)) or in case-based decision (see e.g. Gilboa and Schmeidler (2001)). As far as we know, no specific studies on the axiomatic foundations of these methods for the ranking problem have been proposed. In this perspective, the aim of this paper is to study the descriptive and prescriptive potential of some indirect comparison models using reference points. After showing the interest of introducing reference points in section 2, we will present a basic aggregating model with reference points in section 3, and then we will explore a particular model based on a concordance rule in section 4.

2. Reference dependent preferences

Following Roy (1996), we distinguish three different problems in multicriteria decision making :

- *choice* problem, which consists in choosing one (or several) alternative considered to be the best one for the current situation,
- *ranking* problem, which aims at proposing a complete ranking on the set of considered alternatives,
- *sorting* problem, which consists in classifying the alternatives into predefined categories.

Classical aggregation methods in MCDM to solve these problems are often divided into two different approaches. The first one is based on the use of a value function, as an additive utility (see Keeney and Raiffa (1976), Jacquet-Lagrèze and Siskos (1982)), to compare two alternatives. The second one is based on pairwise comparisons on criteria to determine the preferred alternative, as in the concordance relations (Roy (1996, 1991)).

Multicriteria decision making has strong links with the social choice theory (see Bouyssou et al. (2009) for the similarities and differences between

MCDM and social choice theory). Some fundamental results in social choice theory have a counterpart in MCDM, enabling to highlight the structure of the preferences aggregation in an interesting way. It is particularly obvious in the case of the theorem introduced by Arrow (1951). Transposed in MCDM theory, this result indicates that it is impossible to build a really multicriteria-based aggregation procedure (i.e., without any dictator criterion), leading to a transitive relation, and complying with the principles of universality, unanimity and independence of irrelevant alternatives, when there are more than three criteria and more than three alternatives to be compared. On the other hand, if one of these principles is dropped, new decision rules are going to appear. Some of them have already been investigated: examples of aggregation procedures obtained by weakening one of the contradictory conditions have been proposed by Fishburn (1975, 1976) (lexicographical aggregation procedures) or Weymark (1984) (quasi-transitive preferences). As it is difficult to imagine a multicriteria aggregation procedure that does not satisfy the axiom of non-dictatorship or the axiom of unanimity, the three principles on which we can possibly compromise are transitivity, universality and independence of irrelevant alternatives. Here we focus on the weakening of the independence of irrelevant alternatives axiom. It is possible to obtain complete and transitive preference relations by enabling preferences between two alternatives to depend not only on the considered alternatives, but also on their environment. Particularly, it is possible to compare two alternatives through their behaviour with respect to some third alternatives.

Campbell and Kelly (2000) investigated aggregation procedures using different rules according to the values taken by the alternatives to be compared on a specific criterion. For example, if the value taken by alternative a on criteria i is greater than a specific threshold, then the aggregation rule should be rule 1, and rule 2 otherwise. In a slightly different spirit, we suggest on our side to weaken the axiom of independence with respect to the third alternatives. We enable the preference relation between two alternatives a and b to depend not only on the comparison of these alternatives, but also on their relative positions with respect to one (or several) "reference point".

The presence of reference points in preference relations has been already studied within the framework of strong relations, with, among others, perfectly available information and/or commensurability of the criteria. Several multicriteria optimization methods are based on the use of an ideal point. It can be considered as a reference point, which has to be reached by the alternatives. For example, the TOPSIS method (see Hwang and Yoon (1981)) is based on an ideal and anti-ideal point; the aim of the method is to maximize the distance to the anti-ideal point while minimizing the distance to the ideal point. However, the information available on the alternatives is sometimes poor or incomplete. It is then impossible to have a precise value for each alternative on each criterion, and one often has to deal with an ordinal approach of these values, without any possibility of commensurability or compensation between criteria. Reference points or reference levels are already used in some qualitative multicriteria situations. For example, the MACBETH approach (Bana e Costa et al. (2005)) requires only qualitative judgements about value differences to help an individual or a group quantify the relative attractiveness of options. The MACBETH method uses two fictitious reference levels ("good" and "neutral") to help the decision maker in the evaluation of the alternatives. Reference points or reference levels are also used in sorting problems. In merely qualitative frameworks, several methods have already been proposed to solve sorting problems. A first one is derived from the ELECTRE method and consists in comparing alternatives to reference profiles: this is the object of the ELECTRE TRI sorting method (see Roy (1991), Figueira et al. (2005)). Another approach consists in using rough sets through the dominance-based rough set approach (see Greco et al. (2001b, 2002)), using decision rules to assign the alternatives to the different categories, with respect to some reference levels on each criterion. The axiomatic foundations of the rough set approach have been well studied by Greco, Mattarazo and Slowinski, including characterization of the sorting problem using a utility function or an outranking relation in Greco et al. (2001a) or a Sugeno integral in Slowinski et al. (2002). On another hand, Bouyssou and Marchant (2007a,b) had also proposed a theoretical approach of the ordered sorting problem, through an axiomatic approach of the non-compensatory models using conjoint measurement concepts.

In this paper, we propose to investigate ranking methods inspired by ordered sorting methods, using pairwise comparison with respect to one or several reference points, in the framework of qualitative preference relations.

3. Preference aggregation using reference points: a basic model

3.1. Preliminary definitions and notations

Let us consider a multicriteria decision-making problem characterized by $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$ as a set of alternatives. The set of criteria

$\{1, \ldots, j \ldots, n\}$ is noted N. Each alternative $x \in \mathcal{X}$ is described by the values x_j taken on criteria $j \in N$. $\mathcal{P} = \{p^1, \ldots, p^m\} \subseteq \mathcal{X}$ will denote a set of fixed reference points. We suppose that there exists for all $j \in N$ a binary relation $\succsim_j$ on the elements of $\mathcal{X}_j$, such that $\forall a_j, b_j \in \mathcal{X}_j$, $a_j \succsim_j b_j$ means that the value a_j is considered to be at least at good as the value b_j. We suppose also that relations $\succsim_j$ are complete on $\mathcal{X}_j$.

For any preference relation $\succsim$ on $\mathcal{X}$, $x \succsim y$ means that element x is preferred to element y. The asymmetric part of relation $\succsim$ defines a strict preference relation denoted $\succ$ and defined by: $x \succ y \Leftrightarrow (x \succsim y)$ and not$(y \succsim x)$; its symmetric part defines an indifference relation denoted $\sim$ and defined by $x \sim y \Leftrightarrow (x \succsim y)$ and $(y \succsim x)$.

For any importance relation $\trianglerighteq$ on 2^N, $A \trianglerighteq B$ means that the set of criteria A is considered by the decision maker to be more important than the set of criteria B. The asymmetric part of relation $\trianglerighteq$ defines a strict importance relation denoted $\triangleright$ and defined by: $A \triangleright B \Leftrightarrow (A \trianglerighteq B)$ and not$(B \trianglerighteq A)$; its symmetric part defines an indifference relation denoted $\sim^{\triangleright}$ and defined by $A \sim^{\triangleright} B \Leftrightarrow (A \trianglerighteq B)$ and $(B \trianglerighteq A)$.

We define below pre-orders and complete pre-orders (also named weak-orders) as particular binary relations:

Definition 1. Pre-orders
A binary relation $\succsim$ on $\mathcal{X}$ is said to be a pre-order if

- *$\succsim$ is reflexive on $\mathcal{X}$, i.e., $\forall x \in \mathcal{X}$, $x \succsim x$,*
- *$\succsim$ is transitive on $\mathcal{X}$, i.e., $\forall x, y, z \in \mathcal{X}$, $[x \succsim y$ and $y \succsim z] \Rightarrow x \succsim z$.*

Definition 2. Complete pre-orders
A binary relation $\succsim$ on $\mathcal{X}$ is said to be a complete pre-order if

- *$\succsim$ is a pre-order,*
- *$\succsim$ is complete on $\mathcal{X}$, i.e. $\forall x, y \in \mathcal{X}$, $x \succsim y$ or $y \succsim x$.*

Relations $\succsim_j$ are supposed to be pre-orders on $\mathcal{X}_j$ $\forall j \in N$, but are not supposed to be complete (even if they can be). For any $x \in \mathcal{X}$, we denote X_{p^i} the set $\{j \in N,\ x_j \succsim_j p^i_j\}$. X_p represents the set of criteria where x is considered to be at least as good as reference point p. We suppose although that reference points can be compared to each others on each criterion, i.e. $\forall k, l \in \{1, \ldots, m\}$, $\forall j \in N$, $p^k_j \succsim_j p^l_j$ or $p^l_j \succsim_j p^k_j$.

Relations $\trianglerighteq$ on 2^N are supposed to be complete pre-orders $\forall p \in \mathcal{P}$. They also are supposed to be monotonic with respect to set inclusion, i.e. $\forall A, B \subseteq N$, $A \cup B \trianglerighteq A$.

3.2. Introducing reference levels in preference aggregation model

We propose now a basic model for preference aggregation based on the comparison of the alternatives to the reference points. In order to compare two alternatives x and $y \in \mathcal{X}$ with respect to a set of reference points $\mathcal{P} = \{p^1, \ldots, p^m\}$, we first determine the sets of criteria X_{p^i} and Y_{p^i}, $i = 1, \ldots, m$. Then we compare the two set vectors $(X_{p^1}, \ldots, X_{p^m})$ and $(Y_{p^1}, \ldots, Y_{p^m})$ through a binary relation on $(2^N)^m$ noted $\succsim'$. This relation can be interpreted as a preference relation on the set vectors $(2^N)^m$. This basic model of preference relation w.r.t. reference points, noted model 1, can then be formalized by the following formula:

$$x \succsim y \iff (X_{p^1}, \ldots, X_{p^m}) \succsim' (Y_{p^1}, \ldots, Y_{p^m}) \tag{1}$$

where $\succsim'$ is a binary relation on $(2^N)^m$.

Example 1. *Let us consider a problem with 5 criteria, and a set of reference points $\mathcal{P} = \{p^1, p^2, p^3\}$. A preference relation satisfying model 1 can be as follows: the alternative x is preferred to the alternative y if there is a majority of reference points for which the number of criteria where x is better than the reference point p (noted $|X_p|$) is greater than the number of criteria where y is better than the reference point p (noted $|Y_p|$):*

$$x \succsim y \iff |\ \{p \in \mathcal{P},\ |X_p| \geq |Y_p|\}\ | \quad \geq \quad |\ \{p \in \mathcal{P},\ |Y_p| \geq |X_p|\}\ |\ .$$

We now give a necessary and sufficient condition for a preference relation $\succsim$ to be described by model 1.

Axiom 1. Conditionnal Independance w.r.t. the reference points (CI$\mathcal{P}$)

$$\forall x, y, z, w \in \mathcal{X}, \quad \left[\begin{array}{c} (X_{p^1}, \ldots, X_{p^m}) = (Z_{p^1}, \ldots, Z_{p^m}) \\ (Y_{p^1}, \ldots, Y_{p^m}) = (W_{p^1}, \ldots, W_{p^m}) \end{array} \right] \Rightarrow x \succsim y \iff z \succsim w.$$

This axiom simply means that if two couples of alternatives compare themselves in the same way with respect to every reference points, they must compare themselves in the same way for the global preference relation. Preference relation $\succsim$ depends only on the comparison of alternatives w.r.t. reference points. This axiom is necessary and sufficient to characterize the preference relations satisfying model 1 as stated in the following theorem:

Theorem 1. *Let $\mathcal{P} = \{p^1, \ldots, p^m\}$ be the set of reference points. Then the following conditions 1 and 2 are equivalent:*

1. *preference relation $\succsim$ on $\mathcal{X}$ satisfies axiom CI$\mathcal{P}$,*
2. *there is a binary relation $\succsim'$ on $(2^N)^m$ such that*

$$\forall x, y \in \mathcal{X}, \quad x \succsim y \iff (X_{p^1}, \ldots, X_{p^m}) \succsim' (Y_{p^1}, \ldots, Y_{p^m}).$$

Proof.

$(1 \Rightarrow 2)$ Let us define the relation $\succsim'$ on the m-uplets of sets of criteria such that $A \succsim' B \iff \exists x, y \in \mathcal{X}$ such that $x \succsim y$ and $A = (X_{p^1}, \ldots, X_{p^m})$ and $B = (Y_{p^1}, \ldots, Y_{p^m})$.

Suppose that $z, w \in \mathcal{X}$ are such that $z \succsim w$. By definition of $\succsim'$, we have $(Z_{p^1}, \ldots, Z_{p^m}) \succsim' (W_{p^1}, \ldots, W_{p^m})$.
On the other hand, suppose that there are z, w such that $(Z_{p^1}, \ldots, Z_{p^m}) \succsim' (W_{p^1}, \ldots, W_{p^m})$. So by definition of $\succsim'$, there exist $x, y \in \mathcal{X}$ such that $(X_{p^1}, \ldots, X_{p^m}) = (Z_{p^1}, \ldots, Z_{p^m})$ and $(Y_{p^1}, \ldots, Y_{p^m}) = (W_{p^1}, \ldots, W_{p^m})$ and $x \succsim y$. Then , according to axiom CI$\mathcal{P}$ we have $z \succsim w$.

$(2 \Rightarrow 1)$ Suppose that there is a relation $\succsim'$ on $(2^N)^m$ such that $\forall x, y \in \mathcal{X}, \; x \succsim y \iff (X_{p^1}, \ldots, X_{p^m}) \succsim' (Y_{p^1}, \ldots, Y_{p^m})$. Let x, y, z, w in $\mathcal{X}$ be such that $X_{p^1}, \ldots, X_{p^m}) = (Z_{p^1}, \ldots, Z_{p^m})$ and $(Y_{p^1}, \ldots, Y_{p^m}) = (W_{p^1}, \ldots, W_{p^m})$. If $x \succsim y$, it means that $(X_{p^1}, \ldots, X_{p^m}) \succsim' (Y_{p^1}, \ldots, Y_{p^m})$ and then $(Z_{p^1}, \ldots, Z_{p^m}) \succsim' (W_{p^1}, \ldots, W_{p^m})$. We have then $z \succsim w$, which proves that CI$\mathcal{P}$ stands. $\square$

Remark: Let $\mathcal{P}$ be a given set of reference points $\{p^1, \ldots, p^m\}$ and a preference relation $\succsim$ on $\mathcal{X}$ satisfying axiom CI$\mathcal{P}$. Let us take another set of reference points $\mathcal{Q} = \{q^1, \ldots, q^m\}$, such that:

1. $\forall j \in N$, $q_j^1 \succsim_j q_j^2 \succsim_j \ldots \succsim_j q_j^m$,
2. $\exists n$ permutations of $\{1, \ldots, m\}$ named σ_j, $j = 1, \ldots, n$ such that $q_j^i = p_j^{\sigma_j(i)}$.

As axiom CI$\mathcal{P}$ is satisfied, there exists a binary relation $\succsim'$ on $(2^N)^m$ such that $\forall x, y \in \mathcal{X}, \quad x \succsim y \iff (X_{p^1}, \ldots, X_{p^m}) \succsim' (Y_{p^1}, \ldots, Y_{p^m})$. Let us choose four alternatives x, y, z, and w such that $\forall i = 1, \ldots, m$, $X_{p^i} = Z_{p^i}$ and $Y_{p^i} = W_{p^i}$. By construction of $\mathcal{Q}$, we have also $\forall i = 1, \ldots, m$, $X_{q^i} = Z_{q^i}$ and $Y_{q^i} = W_{q^i}$. So axiom CI$\mathcal{Q}$ is also satisfied, which means that there exists a binary relation $\succsim''$ on $(2^N)^m$ such that $\forall x, y \in \mathcal{X}, \quad x \succsim y \iff (X_{q^1}, \ldots, X_{q^m}) \succsim'' (Y_{q^1}, \ldots, Y_{q^m})$.

This remark indicates that, without any lack of generality, reference points of $\mathcal{P}$ p^1, p^2, ..., p^m can be chosen such that for all $j \in N$, for all $k, l \in [1, \ldots, m]$ with $k < l$, we have $p^k_j \succsim_j p^l_j$. Although this condition is not necessary to validate the following theorems, it will make the proofs easier to state.

4. Axiomatic approach of concordance-based aggregation rules with reference points

Model 1 presents a very general framework for preference aggregation rules w.r.t. reference points. Specific methods conduce to use particular relations $\succsim'$. We present here a particular approach based on the decomposability of relation $\succsim'$ into m relations $\trianglerighteq_p$. Suppose that $\forall p \in \mathcal{P}$, there exists an importance relation $\trianglerighteq_p$ on 2^N. It is then possible to compare x and y w.r.t. each reference point, and so to obtain m preference relations between x and y named $\succsim_p$. Finally we just have to aggregate these various preference relations to obtain global preference relation on the space of the alternatives $\mathcal{X}$. In other words, these approach, denoted model 2, can be modelled by the following formula:

$$x \succsim y \iff \{p \in \mathcal{P} \mid X_p \trianglerighteq_p Y_p\} \succsim_{\mathcal{P}} \{p \in \mathcal{P} \mid Y_p \trianglerighteq_p X_p\} \tag{2}$$

where relations $\trianglerighteq_p$ are importance relations on the subsets of N and $\succsim_{\mathcal{P}}$ is an importance relation on the subsets of $\mathcal{P}$.

Model 2 presents a similarity with the model for concordance relation (Dubois et al. (2003) Bouyssou and Pirlot (2005), Bouyssou and Pirlot (2007)) which generalize the additive model for concordance relation proposed by Roy (1996). A generalized concordance relation is a purely ordinal model for preference relations. It consists in comparing the set of criteria where alternative x is preferred to alternative y (criteria in concordance with the fact that x is preferred to y), and the set of criteria where alternative y is preferred to alternative x (criteria in concordance with the fact that y is preferred to x) through an importance relation on the sets of criteria. It has been formally defined as follows by Dubois et al. (2003):

$$x \succsim y \iff \{j \in N \mid x_j \succsim_j y_j\} \trianglerighteq_N \{j \in N \mid y_j \succsim_j x_j\} \tag{3}$$

where $\trianglerighteq_N$ is an importance relation on the subsets of N, such that $A \trianglerighteq_N B$ means that $A \subseteq N$ is considered to be a set of criteria at least as important as $B \subseteq N$.

We can notice that the introduction of reference points leads to a model based on a generalized concordance relation for comparing two alternatives not directly on their respective criteria values, but on their behaviour w.r.t. each reference point.

4.1. Reference points induce preference relations

Preference relations on $\mathcal{X}$ satisfying model 2 can be characterized by specific properties studied in this section.

First of all, model 2 supposes that there exist m importance relations $\trianglerighteq_p$ on the sets of criteria. It means then that preference relation $\succsim$ between two alternatives $x, y \in \mathcal{X}$ depends only on the sets X_p and Y_p, $p \in \mathcal{P}$.

Let us define for each $p \in \mathcal{P}$ an importance relation $\trianglerighteq_p$ on the subsets of N by the following formula:

$$A \trianglerighteq_p B \iff \exists x, y \mid \begin{cases} X_p = A \\ Y_p = B \\ X_{p'} = Y_{p'} \ \forall p' \in \mathcal{P}, p' \neq p \end{cases} \text{and } x \succsim y. \tag{4}$$

Relations $\trianglerighteq_p$ $p \in \mathcal{P}$ can be seen as m projections of the relation $\succsim$ on the space of subsets of N. Note that if relation $\succsim$ is transitive, then relations $\trianglerighteq_p$, $p \in \mathcal{P}$ are also transitive.

These relations are coherent if there are no couples (x, y) and (z, w) introducing contradictions in the relations defined in formula 4, i.e., couples such as for a reference point $p \in \mathcal{P}$:

$$\begin{cases} X_p = Z_p = A \\ Y_p = W_p = B \\ X_{p'} = Y_{p'} \ \forall p' \in \mathcal{P}, p' \neq p \\ Z_{p'} = W_{p'} \ \forall p' \in \mathcal{P}, p' \neq p \end{cases} \text{and } [x \succsim y \text{ and } w \succ z].$$

This condition is summarized by the following axiom:

Axiom 2. Separability w.r.t. reference points (SEP)
$\forall p \in \mathcal{P}$, $\forall x, y, z, w \in \mathcal{X}$,

$$\left[\begin{array}{lll} X_p = Z_p & X_{p'} = Y_{p'} & \forall p' \neq p \\ Y_p = W_p & Z_{p'} = W_{p'} & \forall p' \neq p \end{array}\right] \Rightarrow [x \succsim y \iff z \succsim w].$$

Axiom SEP is enough to obtain m well-defined importance relations $\trianglerighteq_p$ on the subsets of N. We can then induce m preference relations $\succsim_p$ on $\mathcal{X}$ defined by

$$x \succsim_p y \iff X_p \trianglerighteq_p Y_p.$$

Axiom SEP also induces that for a specific reference point $p \in \mathcal{P}$, the preference relation $\succsim_p$ between two alternatives x and y depends only on the respective sets X_p and Y_p.

Axiom SEP is stronger than axiom CI$\mathcal{P}$ introduced in section 3 as CI$\mathcal{P}$ can be satisfied without SEP being satisfied. We present such a situation in the following example.

Example 2. *Let $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_4$. Let $\alpha_1, \alpha_2, \alpha_3$ be 3 values of $\mathcal{X}_j$ such that $\forall j \in N$, $\alpha_1 \succ_j \alpha_2 \succ_j \alpha_3$. Let $x, y, z, w \in \mathcal{X}$ and p^1, p^2 two reference points such that:*

	p^1	p^2	x	y	z	w
1	α_1	α_2	α_1	α_1	α_2	α_2
2	α_1	α_2	α_2	α_2	α_1	α_1
3	α_1	α_2	α_2	α_3	α_2	α_3
4	α_1	α_2	α_3	α_2	α_3	α_2

We have $X_{p^1} = \{j \in N,\ x_j \succsim_j p^1_j\} = \{1\} = Y_{p^1}$, $Z_{p^1} = \{2\} = W_{p^1}$, $X_{p^2} = \{1, 2, 3\} = Z_{p^2}$, $Y_{p^2} = \{1, 2, 4\} = W_{p^2}$. Note that p^2 states for p in axiom SEP, and p^1 for p'. If $x \succ y$ and $w \succ z$, which is not in contradiction with axiom CI$\mathcal{P}$, then axiom SEP is not satisfied.

4.2. Aggregation of induced preference relations

Axiom SEP implies that preference relations $\succsim_p$ derived from preference relation $\succsim$ following formula 4 are relevant. But it does not ensure in return that preference relation $\succsim$ can be obtained by a specific aggregation method from preference relations $\succsim_p$, as we can see in the following example:

Example 3. *Let $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be 4 values of $\mathcal{X}_j$ such that $\forall j \in N$, $\alpha_1 \succ_j \alpha_2 \succ_j \alpha_3 \succ_j \alpha_4$. Let p^1, p^2, p^3 be 3 reference points, and $a, b, c, d \in \mathcal{X}$ such that:*

	p^1	p^2	p^3	a	b	c	d
1	α_1	α_2	α_3	α_1	α_2	α_1	α_1
2	α_1	α_2	α_3	α_3	α_3	α_2	α_2
3	α_1	α_2	α_3	α_4	α_4	α_3	α_4

We have $A_{p^1} = \{1\}$, $B_{p^1} = \emptyset$, and $A_{p^2} = B_{p^2} = \{1\}$ and $A_{p^3} = B_{p^3} = \{1, 2\}$. Let us suppose that $a \succ b$. By definition of $\trianglerighteq$ proposed in formula 4, it means that $\{1\} \rhd_{p^1} \emptyset$. Similarly, supposing that $c \succ d$ means that $\{1, 2, 3\} \rhd_{p^3} \{1, 2\}$.

Let us now consider the following alternatives $x, y, z, w \in \mathcal{X}$:

	x	y	z	w
1	α_1	α_2	α_1	α_2
2	α_2	α_2	α_3	α_3
3	α_4	α_3	α_4	α_3

According to relations $\trianglerighteq_p$ obtained above, we have:

$$\begin{array}{ll} x \succ_{p^1} y & z \succ_{p^1} w \\ x \sim_{p^2} y & z \sim_{p^2} w \\ x \prec_{p^3} y & z \prec_{p^3} w \end{array}$$

Suppose now that $x \succ y$ and $w \succ z$, which is not contradictory with axiom SEP. We then see that couples (x, y) and (z, w) compare themselves in the same way w.r.t. each of the reference points, but they compare themselves differently for the global preference relation. Preference relation $\succsim$ cannot be thus obtained as a function of the preference relations $\succsim_p$. We have shown that axiom SEP is not sufficient to ensure that preference relation $\succsim$ can be obtained by a specific aggregating method from preference relations $\succsim_p$.

The axiom of conditional independence with respect to the induced relations presented below is thus necessary and sufficient to specify that the preference relation $\succsim$ is obtained by aggregation of the induced relations $\succsim_p$.

Axiom 3. Conditional Independence with respect to the Induced Relations (CIIR)

Let $\succsim$ be a preference relation satisfying axiom SEP and $\{\succsim_p, p \in \mathcal{P}\}$ the induced preference relations as described in formula 4. Preference relations $\succsim$ and $\succsim_p$, $p \in \mathcal{P}$ satisfy the axiom of conditional independence CIIR iff $\forall x, y, z, w \in \mathcal{X}$:

$$\left[\forall p \in \mathcal{P}, \begin{array}{ccc} x \succsim_p y & \iff & z \succsim_p w \\ y \succsim_p x & \iff & w \succsim_p z \end{array}\right] \Rightarrow [x \succsim y \iff z \succsim w].$$

We can now establish the theorem characterizing preference relations satisfying model 2:

Theorem 2. *Let $\succsim$ be a preference relation on $\mathcal{X}$, and $\mathcal{P}$ a set of reference points. Then conditions 1 and 2 are equivalent:*

1. *preference relation $\succsim$ satisfies axiom SEP and axiom CIIR*[1]*,*
2. *there exist m importance relations $\trianglerighteq_p$ on the subsets of N and an importance relation $\succsim_{\mathcal{P}}$ on the subsets of $\mathcal{P}$ such that*

$$x \succsim y \iff \{p \in \mathcal{P} \mid X_p \trianglerighteq_p Y_p\} \succsim_{\mathcal{P}} \{p \in \mathcal{P} \mid Y_p \trianglerighteq_p X_p\}.$$

Proof.

$(1 \Rightarrow 2)$ Let $\succsim$ be a preference relation on $\mathcal{X}$. As axiom SEP is satisfied, there exist m importance relations $\trianglerighteq_p$ on 2^N as defined by formula 4. We define a preference relation $\succsim_{\mathcal{P}}$ on the subsets of $\mathcal{P}$ by the following: let $\mathcal{Q}$ and $\mathcal{Q}'$ be two subsets of $\mathcal{P}$. We say that $\mathcal{Q} \succsim_{\mathcal{P}} \mathcal{Q}'$ iff $\exists x, y \in \mathcal{X}$ such that $x \succsim y$ and $\left\{ \begin{array}{l} \{p \in \mathcal{P} \mid X_p \trianglerighteq_p Y_p\} = \mathcal{Q} \\ \{p \in \mathcal{P} \mid Y_p \trianglerighteq_p X_p\} = \mathcal{Q}' \end{array} \right.$.

Let (z, w) be a couple of $\mathcal{X} \times \mathcal{X}$ such that $z \succsim w$. By definition of $\succsim_{\mathcal{P}}$, we have $\{p \in \mathcal{P} \mid Z_p \trianglerighteq_p W_p\} \succsim_{\mathcal{P}} \{p \in \mathcal{P} \mid W_p \trianglerighteq_p Z_p\}$.

On the other hand, suppose that $z, w \in \mathcal{X}$ are such that $\{p \in \mathcal{P} \mid Z_p \trianglerighteq_p W_p\} \succsim_{\mathcal{P}} \{p \in \mathcal{P} \mid W_p \trianglerighteq_p Z_p\}$. Then by definition of $\succsim_{\mathcal{P}}$, there exists $x, y \in \mathcal{X}$ such that $x \succsim y$ and

$$\left\{ \begin{array}{l} \{p \in \mathcal{P} \mid X_p \trianglerighteq_p Y_p\} = \{p \in \mathcal{P} \mid Z_p \trianglerighteq_p W_p\} \\ \{p \in \mathcal{P} \mid Y_p \trianglerighteq_p X_p\} = \{p \in \mathcal{P} \mid W_p \trianglerighteq_p Z_p\} \end{array} \right.$$

Thanks to axiom CIIR, we have then $x \succsim y \Rightarrow z \succsim w$.

[1]Formally, as CIIR is satisfied, SEP is also satisfied by definition of CIIR. However, for a better understanding, we prefer to recall in the statement of the theorem that relation $\succsim$ has to satisfy sequentially SEP and then CIIR

(2 $\Rightarrow$ 1) Suppose that x, y, z and $w \in \mathcal{X}$ are such that : $X_p = Z_p$, $Y_p = W_p$ and $X_{p'} = Y_{p'}$, $Z_{p'} = W_{p'} \forall p' \neq p$. Then $\{p \in \mathcal{P} \mid X_p \trianglerighteq_p Y_p\} = \{p \in \mathcal{P} \mid Z_p \trianglerighteq_p W_p\}$, $\{p \in \mathcal{P} \mid Y_p \trianglerighteq_p X_p\} = \{p \in \mathcal{P} \mid W_p \trianglerighteq_p Z_p\}$ and so $x \succsim y \iff z \succsim w$, which means that SEP is satisfied.

Suppose that x, y, z and $w \in \mathcal{X}$ are such that : $\forall p \in \mathcal{P}$, $x \succsim_p y \iff z \succsim_p w$ and $y \succsim_p x \iff w \succsim_p z$. It means that $\{p \in \mathcal{P} \mid X_p \trianglerighteq_p Y_p\} = \{p \in \mathcal{P} \mid Z_p \trianglerighteq_p W_p\}$, $\{p \in \mathcal{P} \mid Y_p \trianglerighteq_p X_p\} = \{p \in \mathcal{P} \mid W_p \trianglerighteq_p Z_p\}$ and so $x \succsim y \iff z \succsim w$, which means that CIIR is also satisfied.

$\square$

5. Transitive preference relations and reference points

We study in this section special cases where the preference relation $\succsim$ is transitive. If the relation $\succsim$ also satisfies an unanimity axiom, we show that the only aggregation procedure that ensure that the relation $\succsim$ is transitive is a lexicography of dictator reference points. We first present the notions of unanimity and dictatorship, and deal with the lexicographic order of reference points in a second time.

5.1. Transitivity and dictatorship

Let us define an unanimity property of relation $\succsim$ on $\mathcal{X}$ and a monotonicity property on the set of criteria as follows:

Definition 3. Unanimity (UNA)
A relation $\succsim$ respects unanimity w.r.t. the criteria if:

$$\forall x, y \in \mathcal{X}, \quad [\forall j \in N,\ x_j \succsim_j y_j] \Rightarrow x \succsim y.$$

Definition 4. Monotonicity on the Set of Criteria (MSC)
A relation $\trianglerighteq_p$ is said to be monotone on the set of criteria if $\forall A, B, C, D \subseteq N$, $\forall p \in \mathcal{P}$,

$$A \subseteq B \text{ and } C \subseteq D \Rightarrow [A \trianglerighteq_p D \Rightarrow B \trianglerighteq_p C].$$

Lemma 1. *If $\succsim$ is transitive and satisfies axioms SEP and CIIR, then UNA $\Rightarrow$ MSC.*

Proof. Let $A, B, C, D \subseteq N$ be such that $A \subseteq B$ et $C \subseteq D$. Let $p^k \in \mathcal{P}$, $k \neq m$, be a reference point. As stated in a previous remark, we suppose that $\forall j \in N$, $p^k_j \succsim_j p^{k+1}_j$. Let $x, y, z, w \in \mathcal{X}$ be four alternatives such that:

- $\forall j \in A$, $x_j = p_j^k$, $\forall j \notin A$, $x_j = p_j^{k+1}$
- $\forall j \in B$, $y_j = p_j^k$, $\forall j \notin B$, $y_j = p_j^{k+1}$
- $\forall j \in C$, $z_j = p_j^k$, $\forall j \notin C$, $z_j = p_j^{k+1}$
- $\forall j \in D$, $w_j = p_j^k$, $\forall j \notin D$, $w_j = p_j^{k+1}$

Suppose that $x \succsim w$. It means that $A \trianglerighteq_{p^k} D$ by definition of $\trianglerighteq_{p^k}$. As $A \subseteq B$, we have $\forall j \in N$, $y_j \succsim_j x_j$, and following UNA, we have $y \succsim x$. As $\succsim$ is transitive, $y \succsim w$. So by definition of $\trianglerighteq$, we have $B \trianglerighteq_{p^k} D$. Similarly, UNA implies that $w \succsim z$, transitivity of $\succsim$ implies $y \succsim z$ and by definition of $\trianglerighteq$, $B \trianglerighteq_{p^k} C$, which shows that MSC is satisfied. □

In order to have a framework rich enough to avoid trivial cases, we state for the following a richness axiom RICH, which ensure that we deal with at least 3 really different reference points.

Axiom 4. Richness (RICH)

1. $\forall j \in N$, $\mathcal{X}_j / \sim_j$ *contains at least 4 equivalent classes,*
2. *there are at least 3 criteria* $j \in N$ *and 3 reference points* $p^1, p^2, p^3 \in \mathcal{P}$ *such that* $p_j^1 \succ_j p_j^2 \succ_j p_j^3$,
3. $\exists p^k, p^l \in \mathcal{P}$, $k < l$, *such that* $p^k \succ p^l$,
4. $\exists x, y \in \mathcal{X}$ *such that* $x \succ y$,
5. $\forall p^i \in \mathcal{P}$, $\forall j \in N$, $\exists x \in \mathcal{X}$, $p_j^i \succ_j x_j$.

We now wonder if there is any aggregation procedure of the preference relations $\succsim_p$ which ensure that the global preference relation $\succsim$ is transitive. Following the counterpart of Arrow's impossibility theorem in the field of multicriteria preferences aggregation, the only aggregation procedure which systematically leads to a transitive global preference is a dictatorship of one criterion. It means here that for a given set of reference points $\mathcal{P}$, the only aggregation procedure which leads systematically to a transitive global preference is dictatorship of a reference point $p^* \in \mathcal{P}$, i.e., $\exists p^* \in \mathcal{P}$ such that $\forall x, y \in \mathcal{X}, \quad x \succ_{p^*} y \Rightarrow x \succ y$, as explained in theorem 3. The key-point of the proof of this theorem is the existence of a Condorcet triplet of preference relation $\succsim_p$. In Arrow's theorem, the existence of a Condorcet triplet is ensured by an universality axiom. In our framework, universality on preference relations $\succsim_p$ does not stand as sets X_i and X_j are not independent: if

$i < j$,then $X_i \subseteq X_j$. However, axiom RICH ensure that Condorcet triplet of preferences on relation $\succsim_p$ should appear for a given set of at least 3 reference points as shown in the following example:

Example 4. *We present here an example of situation related to the one needed in the proof of theorem 3. Such a situation can appear as soon as axiom RICH is satisfied. Let $\mathcal{P} = \{p^1, p^2, p^3\}$ and $N = \{1, 2, \dots, k\}$, $k \geq 3$. Each alternative can take its values in the set $\{\alpha_1^j, \alpha_2^j, \alpha_3^j, \alpha_4^j\}$ on criterion j, with $\forall j \in N$, $\alpha_1^j \succ_j \alpha_2^j \succ_j \alpha_3^j \succ_j \alpha_4^j$.*

	1	2	3	$j > 3$
p^1	α_1^1	α_1^2	α_1^3	α_1^j
p^2	α_2^1	α_2^2	α_2^3	α_2^j
p^3	α_3^1	α_3^2	α_3^3	α_3^j
x	α_1^1	α_3^2	α_4^3	α_4^j
y	α_4^1	α_1^2	α_2^3	α_4^j
z	α_2^1	α_3^2	α_2^3	α_4^j

We then obtain the following sets:

	p^1	p^2	p^3
X_{p^i}	$\{1\}$	$\{1\}$	$\{1, 2\}$
Y_{p^i}	$\{2\}$	$\{2.3\}$	$\{2.3\}$
Z_{p^i}	$\emptyset$	$\{1, 3\}$	$\{1, 2, 3\}$

Suppose that $\forall p^i \in \mathcal{P}$,

$$\{1, 2, 3\} \succ'_{p^i} \{1, 2\} \succ'_{p^i} \{2, 3\} \succ'_{p^i} \{1, 3\} \succ'_{p^i} \{1\} \succ'_{p^i} \{2\} \succ'_{p^i} \{3\} \succ'_{p^i} \emptyset$$

Then $\begin{cases} X_{p^1} \succ'_{p^1} Y_{p^1} \succ'_{p^1} Z_{p^1} & \Rightarrow \quad x \succ_{p^1} y \succ_{p^1} z \\ Y_{p^2} \succ'_{p^2} Z_{p^2} \succ'_{p^2} X_{p^2} & \Rightarrow \quad y \succ_{p^2} z \succ_{p^2} x \\ Z_{p^3} \succ'_{p^3} X_3 \succ'_{p^3} Y_{p^3} & \Rightarrow \quad z \succ_{p^3} x \succ_{p^3} y \end{cases}$

Theorem 3. *Let $\mathcal{P}$ be a set of at least 3 reference points. Suppose that preference relation $\succsim$ on $\mathcal{X}$ satisfies axioms RICH. Then if $\succsim$ is a complete preorder satisfying SEP, CIIR, and UNA, then there is a reference point $p^* \in \mathcal{P}$ such that*

$$\forall x, y \in \mathcal{X}, \quad x \succ_{p^*} y \Rightarrow x \succ y.$$

Note that in the frame of this theorem, strict unanimity on the relations $\succsim_p$ is satisfied, as shown in the lemma 2.

Lemma 2. *Let $\mathcal{P}$ be a set of at least 3 reference points. Suppose that preference relation $\succsim$ on $\mathcal{X}$ satisfies axioms RICH. Then if $\succsim$ is a complete preorder satisfying SEP, CIIR, and UNA, then $[\forall p \in \mathcal{P},\ x \succ_p y] \Rightarrow x \succ y$.*

Proof of lemma 2.

Let a and $b \in \mathcal{X}$ be such that $\forall j \in N,\ \forall p \in \mathcal{P},\ a_j \succsim_j p_j$ and $p_j \succ_j b_j$. Element b can be built thanks to axiom RICH, and element a could be p^1 for example. We then have $\forall p^i \in \mathcal{P},\ A_{p^i} = N$ and $B_i = \emptyset$, and so by MSC, $A_{p^i} \trianglerighteq_{p^i} B_i$. Let us show now that $\forall p \in \mathcal{P},\ N \rhd_p \emptyset$. As $\forall i = 1, \dots, n$, $\emptyset \subseteq A_{p^i} \subseteq N$, we have, following MSC, $\forall p^i \in \mathcal{P},\ N \trianglerighteq_{p^i} A_{p^i} \trianglerighteq_{p^i} \emptyset$. If $N \sim^{\rhd}_{p^i} \emptyset$, then, by transitivity, $N \sim^{\rhd}_{p^i} A_{p^i} \sim^{\rhd}_{p^i} \emptyset$, and so $\forall x, y \in \mathcal{X},\ X_{p^i} \sim^{\rhd}_{p^i} Y_{p^i}$. It means that $\forall x, y \in \mathcal{X},\ x \sim_{p^i} y$, which is contradictory with axiom RICH. It is then stated now that $\forall p \in \mathcal{P},\ N \rhd_p \emptyset$. Let us take again alternatives a and b. As $\forall p^i \in \mathcal{P},\ A_{p^i} = N$ and $B_i = \emptyset$, and $\forall p \in \mathcal{P},\ N \rhd_p \emptyset$, it means that $\forall p \in \mathcal{P},\ a \succ_p b$. We know that, by axiom RICH, there exists $p^k, p^l \in \mathcal{P}$ such that $p^k \succ p^l$. As $\forall j \in N,\ a_j \succsim_j p^k_j$, by UNA we know that $a \succsim p^k$. As $\forall j \in N,\ p^l_j \succsim_j b_j$, by UNA we know that $p^l \succsim b$. By transitivity of $\succsim$, we have $a \succsim p^k \succ p^l \succsim b$ and then $a \succ b$. So we have shown that there exist $a, b \in \mathcal{X}$ such that $\forall p \in \mathcal{P},\ a \succ_p b$ and $a \succ b$. By CIIR, it shows that $\forall x, y \in \mathcal{X},\ [\forall p \in \mathcal{P},\ x \succ_p y] \Rightarrow x \succ y$. $\square$

Let us indicate now the detail of the proof of theorem 3. The presentation of the proof is inspired by those of Fishburn (1975) in the frame of multicriteria preference aggregation without any reference point. The key-concept for the demonstration is the decisive set, defined as follow:

Definition 5. *A subset $\mathcal{Q} \subseteq \mathcal{P}$ is said to be decisive for a couple $(x, y) \in \mathcal{X}^2$ if $\{p \in \mathcal{P} \mid x \succ_p y\} = \mathcal{Q}$, $\{p \in \mathcal{P} \mid y \succ_p x\} = \mathcal{P} - \mathcal{Q}$ and $x \succ y$.*

By CIIR, if $\mathcal{Q}$ is decisive for a couple (x, y), it is decisive for all couple $(x, y) \in \mathcal{X}^2$. We say then that $\mathcal{Q}$ is totally decisive.

Proof. As seen in lemma 2, UNA implies that if $\forall p \in \mathcal{P}, x \succ_p y$, then $x \succ y$. It means that $\mathcal{P}$ is decisive. Let K be a minimal (for the inclusion) decisive subset of $\mathcal{P}$.

Suppose that K has more than one element. Let $x, y, z \in \mathcal{X}$, and $p^i \in K$ be such that

$$\begin{array}{ll} & x \succ_{p^i} y \succ_{p^i} z \\ \forall k \in K - \{i\} & y \succ_{p^k} z \succ_{p^k} x \\ \forall j \notin K & z \succ_{p^j} x \succ_{p^j} y \end{array}$$

Such a situation can be easily obtained if axiom RICH is satisfied, as shown in example 4 for a specific set $\mathcal{P}$. Then $y \succ z$ because K is decisive and $\forall p \in K$, $y \succ_p z$. Suppose that $z \succsim x$. Then $y \succ x$ by transitivity, but then K is not a minimal decisive set as for $p^i \in K$, $x \succ_{p^i} y$. So it means that $x \succ z$. But it implies also that $K - \{i\}$ is decisive, which is in contradiction with the fact that K is a minimal decisive set. So K minimal decisive set can only have one element, noted p^*.

We have shown that a minimal decisive set has only one element p^*, which is then unique. We show now that p^* is a dictator, i.e., for all $x, y \in \mathcal{X}$, $x \succ_{p^*} y \Rightarrow x \succ y$. Let $x, y, z \in \mathcal{X}$ be such that $x \succ_{p^*} y \succ_{p^*} z$, and for each $p \neq p^*$, $y \succ_p x$ et $y \succ_p z$. Then $x \succ y$ because $\{p^\alpha\}$ is decisive, $y \succ z$ by unanimity, and $x \succ z$ by transitivity. As $x \succ_{p^*} z$ and as x, z can be chosen such that $x \succ_p z$, $z \succ_p x$ or $x \sim_p z$ is satisfied for all $p \neq p^*$, axiom CIIR implies that for all $x, y \in \mathcal{X}$, $x \succ_{p^*} y \Rightarrow x \succ y$. $\square$

5.2. Lexicographic order on reference points

Theorem 3 says that for a given set of reference points $\mathcal{P}$, each aggregation procedure satisfying the unanimity axiom and giving in every case a transitive global preference relation $\succsim$ by aggregation of the preference relations w.r.t. the reference points $\succsim_p$, $p \in \mathcal{P}$, implies dictatorship of a specific reference point. But what happens if the dictatorial reference point is unable to give a preference between two alternatives? If the preference relation complies with an axiom of strong unanimity on the reference points, it means that the preference relation takes into account the other reference points. So the aggregation procedure is a lexicography of dictatorial reference points, which means that there is a permutation $(.)$ on $\{1, \ldots, m\}$ such that $x \succ y$ iff there is a reference point $p^{(k)}$ such that $(i) < (k) \Rightarrow x \sim_{p^{(i)}} y$ and $x \succ_{p^{(k)}} y$.

In other words, reference point $p^{(k)}$ allows to obtain a preference between alternatives x and y if and only if reference points ranked before it in the lexicographic order see x and y as indifferent. This lexicographic order principle in MCDM has been axiomatised by Fishburn (1975) for direct ordinal aggregation methods. We present in the following theorem a new version including the reference points:

Theorem 4. *Let $\mathcal{P}$ be a set of at least 3 reference points. Suppose that preference relation $\succsim$ on $\mathcal{X}$ satisfies axioms RICH. Then if $\succsim$ is a complete preorder satisfying SEP, CIIR, UNA and a strong unanimity axiom on the preference relations $\succsim_p$ defined by*

$$\forall x, y \in \mathcal{X}, \ \left[\begin{array}{l} \forall p \in \mathcal{P}, \ x \succsim_p y \\ \exists p \in \mathcal{P}, \ x \succ_p y \end{array} \right] \Rightarrow x \succ y.$$

then there exist a permutation (.) on $\{1, \ldots, m\}$, and m preference relations $\trianglerighteq_p$ on the subsets of $\mathcal{P}$ such that

$$\begin{array}{lcl} x \succ y & \iff & X_{p^{(1)}} \succ'_{p^{(1)}} Y_{p^{(1)}} \\ & or & X_{p^{(1)}} \sim'_{p^{(1)}} Y_{p^{(1)}} \ and \ X_{p^{(2)}} \succ'_{p^{(2)}} Y_{p^{(2)}} \\ & \ldots & \\ & or & \forall i \neq m, \ X_{p^{(i)}} \sim'_{p^{(i)}} Y_{p^{(i)}} \ and \ X_{p^{(m)}} \succ'_{p^{(m)}} Y_{p^{(m)}} \\ x \sim y & & otherwise \end{array}$$

Proof. The beginning of the proof is the same as for theorem 3.

Let p^* be the dictatorial reference point found according to theorem 3. We see now alternatives which are not differentiated by the dictatorial reference point, i.e., couples $(x, y) \in \mathcal{X}^2$ such that $x \sim_{p^*} y$. Let us say $p^{(1)} = p^*$ and $\mathcal{P}' = \mathcal{P} - \{p^{(1)}\}$. Consider now $(\mathcal{X}^2)_{\sim_{p^{(1)}}}$ the set of pairs $(x, y) \in \mathcal{X}^2$ such that $x \sim_{p^{(1)}} y$. We then say that the set $K \subseteq \mathcal{P}'$ is decisive in $\mathcal{P}'$ for a pair (x, y) of $(\mathcal{X}^2)_{\sim_{p^{(1)}}}$ if $[\forall p \in K, \ x \succ_p y$ and $\forall p \notin K, \ y \succ_p x]$ implies $x \succ y$. By CIIR, a decisive set for a pair (x, y) should be decisive for all pairs of $(\mathcal{X}^2)_{\sim_{p^{(1)}}}$.

As $\succsim$ satisfies strong unanimity axiom on reference points, $\mathcal{P}'$ is decisive in $\mathcal{P}'$. Let K be a minimal decisive subset in $\mathcal{P}'$ for the inclusion. As seen in proof of theorem 3, we show that the set K contains a singleton noted $p^{(2)}$. This point $p^{(2)}$ is like a dictator on the subset $(\mathcal{X}^2)_{\sim_{p^{(1)}}}$, which means: $\forall x, y \in \mathcal{X}$, $x \sim_{p^{(1)}} y$ and $x \succ_{p^{(2)}} y \Rightarrow x \succ y$. Let us see now the alternatives which cannot be differentiated by reference points $p^{(1)}$ and $p^{(2)}$, i.e., couples $(x, y) \in \mathcal{X}^2$ such that $x \sim_{p^{(1)}} y$ and $x \sim_{p^{(2)}} y$. We then find a third dictatorial reference point on the reduced set, and the proof keeps going on until the last reference point where the unanimity axiom implies the desired conclusion. $\square$

Let us show on an example how does a lexicographic aggregation rule work:

Example 5. *Let the alternatives $x, y, z, w \in \mathcal{X}$ and the reference points p^1, p^2, p^3 be described by the following table, with $\alpha_1 \succ_j \alpha_2 \succ_j \alpha_3 \succ_j \alpha_4$ $j = 1, 2, 3, 4$:*

	p^1	p^2	p^3	x	y	z	w
1	α_1	α_2	α_3	α_3	α_2	α_4	α_3
2	α_1	α_2	α_3	α_2	α_3	α_1	α_2
3	α_1	α_2	α_3	α_3	α_2	α_1	α_1
4	α_1	α_2	α_3	α_1	α_1	α_3	α_3

We want to compare x and y, and z and w. The different subsets X_{p^i} to be considered are the following:

	p_1	p_2	p_3
x	$X_{p^1} = \{4\}$	$X_{p^2} = \{2, 4\}$	$X_3 = \{1, 2, 3, 4\}$
y	$Y_{p^1} = \{4\}$	$Y_{p^2} = \{1, 3, 4\}$	$Y_{p^3} = \{1, 2, 3, 4\}$
z	$Z_{p^1} = \{2, 3\}$	$Z_{p^2} = \{2, 3\}$	$Z_{p^3} = \{2, 3, 4\}$
w	$W_{p^1} = \{3\}$	$W_{p^2} = \{2, 3\}$	$W_{p^3} = \{1, 2, 3, 4\}$

We suppose that the relation $\trianglerighteq_p$ is such that $\mid A \mid \geq \mid B \mid \Rightarrow A \trianglerighteq_p B$.

Suppose that the first reference point in the lexicographic order is p^2. As $Y_{p^2} \triangleright_{p^2} X_{p^2}$, we have $y \succ_{p^2} x$ and then $y \succ x$. If we want to compare z and y, we can see that $Z_{p^2} \sim^{\triangleright}_{p^2} W_{p^2}$ and then $z \sim_{p^2} w$. We have to look at the second reference point in the lexicographic order to determine how z and w compare together.

- *if the lexicographic order is p_2 then p_1 then p_3, then $z \succ w$ because $Z_{p^1} \trianglerighteq_{p^1} W_{p^1}$ and then $z \succ_{p^1} w$.*
- *if the lexicographic order is p_2 then p_3 then p_1, then $w \succ z$ because $W_{p^3} \sim^{\triangleright}_{p^3} Z_{p^3}$ and then $w \succ_{p^3} z$.*

Lexicographic aggregation has strong formal links with the ELECTRE TRI method. ELECTRE TRI method (see Roy (1991, 1996), Figueira et al. (2005), Almeida-Dias et al. (2010), Doumpos et al. (2009)) is a classification method which assign each alternative to an ordered predefined class, using ordinal comparison with specified profiles as boundaries of the classes. After the comparison of the alternative with all the profiles, the next step consists in assigning the alternatives to the categories, for which there are two options: the bottom-up assignment and the top-down assignment. The bottom-up assignment of ELECTRE TRI consists in comparing the alternative to be classified with thresholds raising gradually: the first threshold whom the alternative is not preferred to gives the category where to classify the alternative. On the opposite, top-down assignment of ELECTRE TRI consists in comparing the alternative to be classified with threshold going down: the first threshold which is not preferred to the alternative gives the category where to classify the alternative. The aggregation order on reference points is very similar to the assignment order of ELECTRE TRI. In the same fashion, we can imagine a bottom-up lexicographic aggregation and a top-down lexicographic aggregation. A bottom-up lexicographic aggregation consists in comparing two alternatives beginning by the lowest reference points: it means that an alternative a is preferred to an alternative b because it has less criteria with bad values (the set of criteria where a is preferred to a lower reference point is more important than the similar set for b). A top-down lexicographic aggregation consists in comparing two alternatives beginning by the uppermost reference points: it means that an alternative a is preferred to an alternative b because it has more criteria with good values (the set of criteria where a is preferred to an upper reference point is more important than the similar set for b). In other words, top-down lexicographic aggregation will favour alternatives with very good values on a small number of criteria, and bottom-up lexicographic aggregation will favour alternatives without any bad values on the criteria. We can also take other lexicographic orders: for example with 3 reference points, we can start with comparing alternatives w.r.t. the medium reference point, and then w.r.t. the upper reference point and then w.r.t. the lower reference point.

6. Conclusion

In this paper, we have presented a new approach of ranking problems based on the use of reference points in a purely qualitative framework.

Introducing reference points into MCDA methods enhances the possibility of choosing an adequate method to help the decision maker in its task. We have shown in particular how the introduction of several reference points allows to move on Arrow's theorem and to obtain transitive and non dictatorial preference relations with a lexicographic aggregation.

The proposed model needs only very little information on the alternatives and appears to be a purely qualitative model. The needed information consists in preference relations on each criterion between the alternatives and the reference points (those relations do not need to be complete either). When we compare two alternatives a and b, we do not directly take into consideration whatever a is considered to be better than b on criterion $j \in N$. We only take into account the sets of reference points considered to be not better than a or b on criterion $j \in N$. We then measure the intensity of preference between alternatives, through the use of reference points, only with ordinal information. On another hand, the needed information on the importance relation on the coalition of criteria depends on the complexity of the model: basically, a lexicography only needs to know the order on the reference points.

Analysing real-life preference relations, the introduction of reference points in an ordinal multicriteria aggregation rule also enables to model preference relations which can not be described by an approach only taking into account the direct comparison between two alternatives. Elicitation procedures should be used in real-case decision problems to determine the parameter values of the model. The main issue in the elicitation procedure is that parameters as reference levels, the importance relation on the sets of criteria, and the importance relation on the sets of reference points are not independent. Several parameter combinations often enable to represent the considered preference relations. It is then possible to assume that some parameters are already fixed. For example, reference points should be considered as primitives for the model, as developed in this paper, if the decision-maker keeps well-known reference points in mind. It is also possible to assume that reference points are considered as parameters for the model, and have to be revealed. One can also determine only the number of required reference points. The main issue is then that there often exist several suitable reference points to represent the preference relations, especially when the values taken by the criteria follow continuous scales. These elicitation issues will be developed in a forthcoming paper.

Acknowledgment

The author wants to thank anonymous referees for their very helpful comments on earlier versions of this paper.

Bibliography

Almeida-Dias, J., Figueira, J.R., Roy, B., 2010. Electre Tri-C: A multiple criteria sorting method based on characteristic reference actions. European Journal of Operational Research 204, 565–580.

Arrow, K.J., 1951. Social choice and individual values. Cowles Foundations and Wiley, New York.

Bana e Costa, C.A., De Corte, J.M., Vansnick, J.C., 2005. On the mathematical foundations of MACBETH, in: Figueira, J., Greco, S., Ehrgott, M. (Eds.), Multiple Criteria Decision Analysis: State of the Art Surveys. Springer Verlag, Boston, Dordrecht, London, pp. 409–443.

Bouyssou, D., Marchant, T., 2007a. An axiomatic approach to noncompensatory sorting methods in MCDM, I: The case of two categories. European Journal of Operational Research 178, 217–245.

Bouyssou, D., Marchant, T., 2007b. An axiomatic approach to noncompensatory sorting methods in MCDM, II: More than two categories. European Journal of Operational Research 178, 246–276.

Bouyssou, D., Marchant, T., Perny, P., 2009. Social choice theory and multicriteria decision aiding, in: Bouyssou, D., Dubois, D., Pirlot, M., Prade, H. (Eds.), Decision-making Process Concepts and Methods, ISTE & Wiley. pp. 779–810.

Bouyssou, D., Pirlot, M., 2005. A characterization of concordance relations. European Journal of Operational Research 167, 427–443.

Bouyssou, D., Pirlot, M., 2007. Further results on concordance relations. European Journal of Operational Research 181, 505–514.

Campbell, D.E., Kelly, J.S., 2000. Information and preference aggregation. Social Choice and Welfare , 3–24.

Doumpos, M., Marinakis, Y., Marinaki, M., Zopounidis, C., 2009. An evolutionary approach to construction of outranking models for multicriteria classification: The case of the ELECTRE TRI method. European Journal of Operational Research 199, 496–505.

Dubois, D., Fargier, H., Perny, P., Prade, H., 2003. A characterization of generalized concordance rules in multicriteria decision making. International Journal of Intelligent Systems, special issue on preference modeling application , 751–774.

Figueira, J., Mousseau, V., Roy, B., 2005. ELECTRE methods, in: Figueira, J., Greco, S., Ehrgott, M. (Eds.), Multiple Criteria Decision Analysis: State of the Art Surveys. Springer Verlag, Boston, Dordrecht, London, pp. 133–162.

Fishburn, P.C., 1975. Axioms for lexicographic preferences. Review of economic studies 42, 415–419.

Fishburn, P.C., 1976. Noncompensatory preferences. Synthese 33, 393–403.

Gibbard, A.F., 1973. Manipulation of voting schemes: A general result. Econometrica 41, 587–601.

Gilboa, I., Schmeidler, D., 2001. A Theory of Case-based decisions. Cambridge University Press.

Greco, S., Matarazzo, B., Slowinski, R., 2001a. Conjoint measurement and rough set approach for multicriteria sorting problems in presence of ordinal criteria, in: Colorni, A., Paruccini, M., Roy, B. (Eds.), A-MCD-A, Aide Multicritère à la Décision/Multiple Criteria Decision Aid. European Commission, Joint Research Centre, EUR 19808 EN, Ispra, pp. 117–144.

Greco, S., Matarazzo, B., Slowinski, R., 2001b. Rough sets theory for multicriteria decision analysis. European Journal of Operational Research 129, 1–47.

Greco, S., Matarazzo, B., Slowinski, R., 2002. Rough sets methodology for sorting problems in presence of multiple attributes and criteria. European Journal of Operational Research 138, 247–259.

Hwang, C.L., Yoon, K., 1981. Multiple attribut decision making : Methods and applications : a state-of-the-art survey. Springer-Verlag.

Jacquet-Lagrèze, E., Siskos, Y., 1982. Assessing a set of additive utility functions for multicriteria decision making: the UTA method. European Journal of Operational Research 10, 151–164.

Keeney, R.L., Raiffa, H., 1976. Decisions with multiple objectives: Preferences and value tradeoffs. J. Wiley, New York.

Roy, B., 1991. The outranking approach and the foundations of ELECTRE methods. Theory and Decision 31, 49–73.

Roy, B., 1996. Multicriteria Methodology for Decision Aiding. Kluwer Academic Publisher.

Satterthwaite, M., 1975. Strategy proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory 10, 187–217.

Slowinski, R., Greco, S., Matarazzo, B., 2002. Axiomatization of utility, outranking and decision-rule preference models for multiple-criteria classification problems under partial inconsistency with the dominance principle. Control and Cybernetics 4, 1005–1035.

Weymark, J.A., 1984. Arrow's theorem with social quasi-orderings. Public Choice , 235–246.