Masters DSC/MLDM/CPS2

Optimization & Operational Research - Exam

(26/03/2019) 2h00 : personal documents allowed

Exercise 1 : A sum of two functions (8 points)

The aim of this exercise is to study a function h which is defined as the sum of two other functions : f and g. The two first parts focus on th study of f and g and last part on h on which different gradient descent algorithms are applied.

We consider two parameters $\gamma, \delta \in \mathbb{R}$. The aim of this exercise is to study function $h_{\gamma,\delta} : \mathbb{R}^4 \to \mathbb{R}$ defined by :

$$h_{\gamma,\delta}(x_1, x_2, x_3, x_4) = f_\gamma(x_1, x_2) + g_\delta(x_3, x_4),$$

where $f_\gamma, h_\delta : \mathbb{R}^2 \to \mathbb{R}$ are defined by :

$$f_\gamma(x_1, x_2) = \frac{1}{2}\left(x_1^2 + 8x_2^2 - 2\gamma x_1 x_2\right) + x_1 - x_2 + 3,$$
$$h_\delta(x_3, x_4) = \delta x_3^3 + \frac{1}{2}\left(2x_3^2 + x_4^2 + x_3 x_4\right) - 6x_3 + 2x_4.$$

Part A : Study of f_γ

This part focuses on the function f_γ. The aim is to find the nature of the extrema of this function according to the paramter γ.

1. Study the convexity of the function f_γ.
2. Give the solution of *Euler's Equation*, i.e. the solution of the linear system $\nabla f_\gamma(x, y) = (0, 0)$, for all values of $\gamma \notin \{-2\sqrt{2}, 2\sqrt{2}\}$.
   What happens when $\gamma = -2\sqrt{2}$ or $\gamma = 2\sqrt{2}$?
3. Give the nature of the previous extrema (i.e. say if it is local/global minimum/maximum of the function). The nature of the extremum depends on γ.

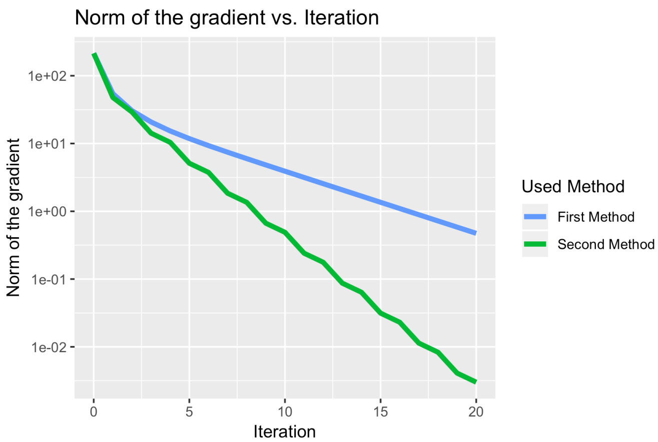
Part B : Study of g_δ

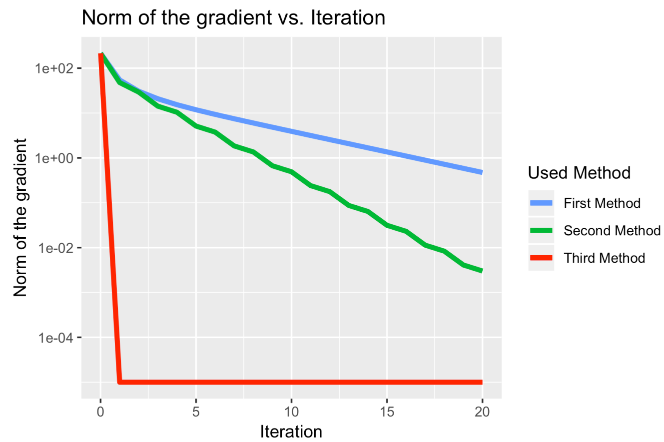
This part focuses on the function g_δ. The aim is to find the nature of the extrema of this function according to the paramter δ.

1. Study the convexity of the function g_δ.
2. Give the solution of *Euler's Equation*, i.e. the solution of the linear system $\nabla g_\delta(x, y) = (0, 0)$, for all values of δ.
3. Give the nature of the previous extrama (i.e. say if it is local/global minimum/maximum of the function). The nature of the extremum depends on δ.

Part C : Minimization of $h_{\gamma,\delta}$

1. Show that the sum of two convex functions is convex.
2. Using the previous parts, show that the function $h_{\gamma,\delta}$ is convex if and only if $\gamma \in [-2\sqrt{2}, 2\sqrt{2}]$ and $\delta = 0$.
3. Suppose that $\gamma \in [-2\sqrt{2}, 2\sqrt{2}]$ and $\delta = 0$. Using the previous parts, what is the global minimum of the function $h_{\gamma,\delta}$?
4. In the following, we consider that both γ and δ are equal to 0. Thus, the function $h_{0,0}$ is convex. The figure below on the left illustrates the convergence of two variations of gradient descent algorithm : (i) *gradient descent with a fixed step* and (ii) *gradient descent with an optimal step*





   (a) Recall what the gradient descent algorithm consists of in general.
   (b) Explain which curve corresponds to which method. Justify your answer.
   (c) The figure on the right illustrates another variation of the gradient descent algorithm represented by the red curve (mentioned in class). In your opinion, which algorithm is that ? (Bonus)
   (d) If you have found the right algorithm, try to explain why it converges in only one iteration in this case. (Bonus+)

Exercise 2 : Linear programming (6 points)

Part A : Formulate and solve an optimization problem

A chemical firm makes two types of industrial solvents, S_1 and S_2. Each solvent is a mixture of three chemicals. Each kL of S_1 requires 12L of chemical **A**, 9L of chemical **B**, and 30L of chemical **C**. Each kL of S_2 requires 24L of chemical **A**, 5L of chemical **B**, and 30L of chemical **C**. The profit per kL of S1 is \$100, and the profit per kL of S2 is \$85. The inventory of the company shows 480 L of chemical **A**, 180 L of chemical **B**, and 720 L of chemical **C**. Assuming the company can sell all the solvent it makes, find the number of kL of each solvent that the company should make to maximize profit.

1. Formulate the corresponding optimization problem and explain the meaning behind the introduced variables ;
2. Do one step of the simplex algorithm. Explain which is the entering / leaving variable and why. Give the tableau before and after this step ;
3. Finally, give the new feasible solution and explain what will happen next (e.g., this is the optimal solution, there is no optimal solution, ...).

Part B : Understanding simplex method

1. What are the basic variables of the problem given below ? What is the value of nonbasic variables ?

B			X1	X2	X3	X4	X5
X ?	4	=	0	1/3	-1	1	0
X ?	20	=	1	2/3	-1	0	0
X ?	80	=	0	-1	3	0	1
	Z-3	=	0	-25/3	0	40/3	0

2. What can you say about the solution of the following minimization problem ?

B			X1	X2	X3	X4
X3	24	=	0	1	1	0
X4	20	=	-2	2	0	1
	Z-0	=	-40	-35	0	0

Exercise 3 : Constrained optimization (6 points)

Consider the following constrained optimization problem with an arbitrary constant c_1 :

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & c_1x_1 - 4x_2 - 2x_3 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 2 \\ & x_2^2 + x_3^2 \leq 2 \\ & x_1^2 + x_3^2 \leq 2 \end{aligned}$$

1. Write down the KKT-conditions for the problem.
2. Are there any values for the constant c_1 which make the point $\mathbf{x} = (1.4, 0.2, 0.2)^T$ an optimal solution to the problem ? If there are any such values, determine all of these values for c_1.
3. Reply to the previous question with $\mathbf{x} = (1, 1, 1)^T$.
4. Assume that $c_1 = -6$. Compute the value of the dual objective function $g(\lambda)$ in the point $\lambda^* = (1, 1, 1)^T$, and determine whether it is an optimal solution to the dual problem.